# ELASTIC AND VISCOELASTIC SANDWICH PLATES

By

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To my parents

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# KEY TO SYMBOLS

$\gamma_{ij}$ = Cartesian strain tensor

$V_i$ = Components of the displacement vector referred to the coordinates of the undeformed body

${}_{o}V$, $dV$ = Volume of the undeformed body and related element of volume respectively

${}_{o}A_{(t)}$, ${}_{o}A$, ${}_{o}A_{s(t)}$, ${}_{o}A_{v(t)}$ = Total surface area, area of the middle surface, total surface area over which stresses are prescribed and total surface area over which displacements are prescribed, respectively, of the undeformed body

${}_{o}A_{s(U)}$, ${}_{o}A_{s(L)}$, ${}_{o}A_{V(U)}$, ${}_{o}A_{v(L)}$ = Portion of the upper surface and the portion of the lower surface over which stresses are prescribed, portion of the upper surface and portion of the lower surface over which displacements are prescribed

$dA$ = Element of area

${}_{o}A_{cs}$, ${}_{o}A_{cv}$, $ds$ = Portion of the edge surface over which stresses are given, portion of the edge surface over which displacements are given and related element of length along the middle of the edge respectively

$\delta(\ )$ = Variation of a quantity

$\rho_o$ = Density of the undeformed body

${}_of_i$, ${}_oF_i$ = Components of acceleration and components of body force respectively, referred to the coordinates of the undeformed body

$S_{ij}$ = Second Piola-Kirchhoff stress tensor as defined in [50]

$\varepsilon^*$ = Strain energy per unit volume of the body

$\tilde{(\ )}$ = Prescribed quantities

$S_{i*}$ = Components of the surface traction vector referred to the coordinates of the undeformed body

${}_on_i$ = Unit normal to the undeformed surface

$\delta_{ij}$ = Kronecker delta

$2h$, $t_f$, $t_c$ = Thickness of the whole plate, the face and the core respectively of a sandwich plate

$x_i$, $x'_i$, $x''_i$ = Cartesian coordinates for the core, the upper and lower faces respectively as shown in Figure 2.

$U_i$, $U'_i$, $U''_i$, $\Psi_i$, $\Psi'_\alpha$, $\Psi'_3$, $\Psi''_\alpha$, $\Psi''_3$ = Displacement functions defined in (14)

$N_{\alpha\beta}$, $N'_{\alpha\beta}$, $N''_{\alpha\beta}$, $M_{\alpha\beta}$, $M'_{\alpha\beta}$, $M''_{\alpha\beta}$, $Q_\alpha$, $Q'_\alpha$, $Q''_\alpha$, $K_{\alpha\beta}$, $T_\alpha$, $N_{33}$ = Stress resultants in different layers of a sandwich plate as defined in (24)

$f_\alpha$, $f'_\alpha$, $f''_\alpha$, $F_\alpha$, $F'_\alpha$,
$F''_\alpha$, $m_\alpha$, $m'_\alpha$, $m''_\alpha$, $M_\alpha$,
$M'_\alpha$, $M''_\alpha$ = Body force and acceleration resultants of the different layers of a sandwich plate as defined in (24)

$\tilde{S}_i$, $\tilde{S}'_i$, $\tilde{S}''_i$, $\tilde{t}_i$,
$\tilde{t}'_\alpha$, $\tilde{t}''_\alpha$ = Given stress resultants at the edges as defined in (24)

${}_n\gamma_{\alpha\beta}$, ${}_n\gamma'_{\alpha\beta}$,
${}_n\gamma''_{\alpha\beta}$ = Strain functions as defined in (30)

$\varepsilon$, $\varepsilon'$, $\varepsilon''$ = Strain energy per unit area of the core, the upper and lower surfaces respectively

$\nu_c$, $\nu_f$ = Poisson's ratio of the core and the faces respectively

$G_c$, $G_f$ = Shear moduli of the core and the faces respectively

$\overline{N}_\alpha$, $N^\circ_{\alpha\beta}$ = Axial stress resultants in the core and in the faces respectively prior to buckling

$n_{\alpha\beta}$, $n'_{\alpha\beta}$,
$n''_{\alpha\beta}$ = Axial stress resultants due to small transverse deflection

w, W = Antisymmetric and symmetric deflections of a sandwich plate

$\underline{p}'$, $\underline{p}''$ = Transverse loads per unit area

N = Uniformly distributed axial compressive load in the x-direction

$a, b$ = Dimensions of the plate in x and y directions respectively

$\tilde{q}, \hat{q}$ = $\frac{1}{2G_f t_f}(p' \pm p'')$

$g, r, f$ = $\frac{G_c t_c}{G_f t_f}, \quad \frac{t_f}{t_c}, \quad \frac{N}{G_f t_f}$

$\alpha_f, \alpha_c$ = $\frac{1}{1 - \nu_f}, \quad \frac{2(1 - \nu_c)}{1 - 2\nu_c}$

$\bar{F}, \bar{\bar{F}}$ = Laplace transform and s-multiplied Laplace transform respectively of a quantity F(t)

$Y_c, Y_f$ = Relaxation moduli for shear of the core and the faces respectively

$t$ = Time

$N_{cr(a)}, N_{cr(s)}$ = Critical values of N corresponding to the antisymmetric and the symmetric modes respectively of buckling

Symbols not listed here have been defined at the time of their first usage.

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# ELASTIC AND VISCOELASTIC SANDWICH PLATES

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A nonlinear theory of elastic sandwich plates has been developed from the Modified Hellinger-Reissner principle in three-dimensional elasticity using a variational method. A consistent method of reducing in steps the most general three-dimensional mixed boundary value problem to, first, a two-dimensional mixed boundary value problem, and then to a single layer or multi-layer plate problem by assuming different displacement functions, has been presented. A complete set of equilibrium equations, boundary conditions, compatibility equations and strain functions in terms of displacement functions is derived in a unified approach from the variational principle mentioned above, whereas in the earlier theories of other authors, the compatibility equations and the relations between strain functions and displacement functions had to be considered separately and a priori.

The existing analysis of an elastic sandwich plate subjected to combined axial and transverse loads and the corresponding buckling problem are shown to be special cases of the general analysis presented here.

In the second part of this dissertation, a known solution of the problem of a rectangular sandwich plate, simply supported on all sides and subjected to arbitrary transverse loads on both surfaces and uniformly distributed uniaxial loads on two opposite edges, has been used to get the solution of a similar problem of a sandwich plate made of viscoelastic materials, using the correspondence principle of viscoelasticity theory. Expressions for time dependent antisymmetric and symmetric deflections and time dependent critical values of the compressive axial loads for both antisymmetric and symmetric modes of buckling have been derived. Calculations have been done and results have been shown for some specific problems.

# CHAPTER 1

# INTRODUCTION

## General Background

A sandwich plate is the plate form of structural sandwich construction which is defined [1]* by the American Society for Testing and Materials as:

> A laminar construction comprised of alternate dissimilar simple or composite materials assembled and intimately fixed in relation to each other so as to use the properties of each to attain specific structural advantages for the whole assembly (p. 1.3).

A sandwich structure generally consists of relatively thin external members called facings separated by and bonded to a relatively thick low density internal member called a core. Usually the facings are composed of high strength materials and the core is made of materials having low average density and strength.

Sandwich structures exhibit the following advantages: high ratio of bending stiffness to weight, resistance to fatigue, sound damping, thermal insulation and aerodynamically smooth surfaces.

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*Numbers in brackets refer to Bibliography.

Continuous improvements in the techniques and technology of adhesive bonding have made the adhesive-bonded sandwich structures practical in an ever-increasing range of applications. Examples are:

Aircraft: Control surfaces, control tabs, bulkheads, helicopter blades, flooring, doors, engine cowlings, wing panels, fairings, fan air ducts, etc.

Missile and spacecraft: Fins and control surfaces, tanks, structural shells, heat shields, intertank structures, etc.

Electronics and communication: Antenna reflectors, radomes, electronic packaging, etc.

Building: Pre-fabricated shelters, curtain walls, partitions, doors, etc.

Miscellaneous: Desk top scaffolding, house trailer flooring, boats, cargo containers, etc.

Depending on the conditions under which it is used, the material of the faces may be an aluminum alloy, reinforced plastic, titanium, heat-resistant steel, etc. A very popular type of core now in use is the "honeycomb" core, which consists of thin foils in the form of hexagonal cells perpendicular to the faces. Other types of core are corrugated sheet, expanded materials like cellulose acetate, synthetic rubber, balsa wood, etc. The adhesive material used for bonding the layers to the core may be Nitrile Phenolic, Vinyl Phenolic, Epoxy Phenolic, Unmodified Epoxy, Epoxy Polymide, etc. The dimensions of a sandwich construction vary widely from one case to another. A typical ratio of core thickness to the face thickness may be 20.

Although there were some earlier applications, sandwich construction aroused great interest only after the Second World War when laminar-flow surface was considered to be an extremely desirable design feature for high speed aircraft and when it became known that the famous Mosquito airplane was constructed in this manner. Therefore, the sandwich construction is a comparatively new branch of structural elements with a history of about 30 years. But because of its wide range of applicability, it has attracted the attention of hundreds of investigators from all over the world, the sum total of whose contributions has established its technology on a highly refined and ever-improving theoretical basis. The total number of articles published in this field (well over 1,000 articles and more than 10 books) also signifies the importance of sandwich construction.

The theory and analysis of the sandwich plate are much more involved than and different from those of ordinary single layer plates because the response of sandwich construction to various inputs is determined to a large extent by the geometrical and material properties of the core relative to the facings. The single layer plate can be considered to be a special case of the sandwich plate and the analysis of the former can be derived as a special case from, and be used as a simple check on, the analysis of the latter.

A general survey of the extensive literature on sandwich plates provides the history and the present state of the art which cover the effect of transverse shear deformation on the small as well as large deflections of sandwich plates, the local instability due to wrinkling of the facings, the intracell buckling of honeycomb sandwiches, the general instability of sandwich structures, the thermal effects as well as the problems of dynamic response including the effects of various inertia terms, damping, and large amplitudes. A few reports on viscoelastic behavior and wave propagation problems of sandwich plates are also available.

In the following, an attempt is made to recount the most prominent phases of development of the theory of sandwich plates while, for fuller surveys, it is suggested that the reader see the comprehensive bibliographies [2], [3] and the review articles [4] and [5].

## History and Present State of the Art

### Small Deflection Theory

The early theory, which accounts for the transverse shear effect in the core by assuming that a linear element initially straight and normal to the middle plane of the core will remain straight after deformation but will deviate from the normal to the deformed middle plane by an amount represented by a parameter (which depends on the elastic properties of the material), was developed by a number of authors, e.g. [6], [7]. The later theory of Libove and Batdorf [8] and Reissner [9] has led to the deflection equations of sandwich plates on the basis of a simplified model. Reissner [10] noted that, for such simplifying assumptions, the equilibrium equations of a sandwich plate have the same form, except for constants, as his equations for homogeneous plates with transverse shear deformation taken into account. Various effects previously omitted have been incorporated into the analysis of Eringen [11] who generalized the variational approach used by Hoff [12]. A comparatively recent attempt by Mushtari [13] has resulted in considerable refinement of earlier theories.

Thermal effects in sandwich plates have been investigated by Egcioglu [14], among others, with the help of suitable transformations of the variables in order to obtain two independent systems of differential equations. More recent works including [15] and [16] show minor improvements of the theory.

## Large Deflections

Reissner [17] employed a simplified model consisting of two facings acting as membranes and a core resisting transverse shear and normal stresses only for deriving a set of equations governing the large deflection of sandwich plates. The principle of complementary energy has been used by Teichmann and Wang [18] for curved sandwich plates and cylinders in deriving non-linear stress resultant-displacement relations.

Several contributions to the large deflection analysis of sandwich structures have been published in the Soviet Union over the past fifteen years. Many of these have been summarized by Habip [4]. They contain analyses of orthotropic cores using both the variational and the classical methods.

Recently Ebcioglu [19], [20] has given a non-linear theory of sandwich panels in the reference state using Hamilton's principle. He obtained the equations of motion and the boundary conditions for twelve displacement functions. The analysis has included orthotropic materials. Numerical solutions of specific problems have been worked out by many authors, e.g. [21].

## General Buckling

Of all the earlier work done in this field, the one by Eringen [22] stands as the most prominent, and it gives solutions for both the antisymmetric and symmetric modes of buckling. The work of Hoff [12] is also worth mentioning in this context. However, all these works were associated with simple boundary conditions. Nardo [23] obtained an exact solution for a rectangular sandwich plate with loaded edges clamped and the other two edges simply supported. The energy method has been used by Ericksen and March and they published a series of papers [24], [25], [26] on the solution of problems for a number of useful boundary conditions. In their solutions, the materials of both the faces and the core may be orthotropic. An improvement of the Ericksen-March method is given in [27], which gives solution for the buckling by uniaxial compressive load. Moskalanko [28] obtained linearized equations of non linear elasticity and also found the components of the stress tensor perturbations across the plate.

Among the recent reports on experimental determination of the critical stresses for buckling, the one by Zender [29] gives results for a large number of cases. He has compared his results with the theoretical results of orthotropic plates. The comparison shows that the open-face truss-core sandwich panel is substantially less efficient than indicated by theory. The inefficiency was perhaps the result of reduced stiffness near the edges.

Numerical results have been obtained by Benson [30] and Landgren and Salama [31] on the solution of buckling problems. Benson considered both sandwich plates and columns with the effects of shear and normal stresses in the core which is orthotropic. Using variational methods, he obtained differential equations governing the phenomena of both general buckling and face wrinkling. Landgren and Salama used finite element methods to solve the problem of buckling of multilayer plates. Ebcioglu and Kim [32] modified some of the assumptions made by earlier authors and solved both the antisymmetric and the symmetric problems of buckling of rectangular sandwich plates making use of the continuity relations at the interfaces between layers.

## Local Buckling

Of the earlier reports on the problem of local buckling, Norris [33] and Eringen [22] presented relatively simple methods for finding the wrinkling stresses in the facing of an orthotropic sandwich construction. Yusuff's [34] theory took into consideration the initial imperfections of sandwich plates. A different type of local instability due to the particular geometry of the core, such as honeycomb, is intracell buckling of the facings. Weikel and Kobayashi [35] have reported some results obtained under various simplifying assumptions and compared them with the earlier attempts of Norris and Kommers [36]. Recently, Benson [30] has presented both theory and numerical results based on variational principles. They also considered general buckling as mentioned earlier.

Apart from the areas described above, vibration problem of sandwich plates has been treated by several authors including Yu [37], [38] who has investigated the simple-shear modes of vibration of sandwich plates on the basis of a comprehensive sandwich plate theory and also the nonlinear flexural vibrations of sandwich plates. The high frequency extensional vibrations of sandwich plates have been investigated by Chu [39]. The wave propagation problem in sandwich plates has been investigated by Armenakas [40], among others.

### Previous Work Related to and the Objectives of This Dissertation

As mentioned earlier in this section, Ebcioglu [19], [20] obtained a complete set of equations of motion and boundary conditions from Hamilton's principle using the variational method. However, the equations relating various stress resultants to strain functions and the relations between strain functions and displacement functions had to be considered separately and a priori. Habip [41] used a different procedure. He started with the Modified Hellinger-Reissner's Principle and obtained the relations between stress resultants and strain functions, and between strain functions and displacement functions, in addition to the equations of motion and the boundary conditions, in a unified approach for a single layer plate. A formulation similar to Habip's for a general nonlinear sandwich plate problem is non-existent in the literature. Therefore, it seems both timely and important to make an attempt to fill this void. The analysis presented in the first part of this dissertation (Chapter 2) has been devoted to making such an attempt.

Chapter 3 contains a derivation of the buckling equations by simplifying the results obtained in Chapters 2 and 3. These simplified equations have been obtained by Kim [42] who has considered the problems of combined axial and transverse loads, and buckling of a simply supported rectangular sandwich plate. Thus, it has been shown that Kim's equations can be obtained as a special case of the general problem treated in Chapters 2 and 3.

Though the elastic bending and elastic stability problems of sandwich plates have received the attention of numerous investigators, the viscoelastic properties of the materials have been neglected. As mentioned in the first section of this chapter, the facings are often made of reinforced plastic and the core material can be an expanded material such as cellulose acetate, synthetic rubber, etc. All these materials exhibit viscoelastic behavior which may considerably influence the bending and stability of sandwich plates. The author could find very few papers on the viscoelastic behavior of sandwich plates, and almost all of them are very recent. This suggests that the importance of this aspect of sandwich plates has just begun to be properly realized.

Chang and Fang [43] made a valuable contribution by investigating the flexural motion of a rectangular sandwich plate with elastic facings and a viscoelastic core. The dynamic response to a unit step function was calculated first and then the general transient and steady-state responses to an arbitrary input function were expressed in terms of the unit response by means of convolution integral. Malmstrom [44] and Tang [45] analyzed the problems of creep bending of sandwich plates and two layer plates respectively, but their theories are of plasticity. Ashton and Moavenzadeh [46] considered a three-layer viscoelastic half space and analyzed the stress distribution in each layer. However, the analysis given by

Ashton is too general (perhaps because of his interest in Civil Engineering) to be of any significance in sandwich plate theory. Also, none of the authors mentioned in this paragraph considered the problem of instability.

The only paper that gives an analysis of bending and buckling is by Lisy [47] who considered the problem of a rectangular simply supported viscoelastic sandwich plate subjected to a uniaxial compressive load and a uniformly distributed transverse load. However, several assumptions have been made by Lisy [47]. These are:

(a) Only the core is viscoelastic; the faces are elastic.

(b) Transverse strain in the core and, hence, the possibilities of the symmetric deflection and buckling are neglected.

(c) Shear strains in the facings are very small compared to those in the core.

(d) The material of the core exhibits viscoelastic behavior governed by linear Maxwell-Thomson differential equation.

(e) The deflection of the plate is a single sine function with a half wave length equal to the length of the plate in each x and y direction.

In addition to the above limitations, the critical value of the compressive load for buckling was calculated by Lisy only at time = infinity. Thus the applicability of the result obtained by Lisy is severely limited in scope.

In this dissertation, an attempt has been made to analyze a similarly loaded sandwich plate but without making any of the assumptions listed above, except assumption (d). Linearity in the material response function has been retained as the first order approximation to the actual case, while a better and more complicated model will perhaps be the subject of future investigations.

Chapter 4 gives the analysis of viscoelastic sandwich plates derived from elastic sandwich plates using the correspondence principle, and Chapter 5 gives the numerical results and conclusions.

# CHAPTER 2

# GENERAL NONLINEAR ANALYSIS OF ELASTIC PLATES

## Notation and Coordinates

Throughout this dissertation, Cartesian coordinates have been used. In most places, especially in the first part of this dissertation where long expressions are involved, indicial notation has been used for the purposes of compactness and convenience. If the components of a vector $\bar{a}$ in the x, y, z coordinates are $a_x$, $a_y$, $a_z$ respectively, in indicial notation these will be denoted by $a_i$, $i = 1, 2, 3$. Thus, $a_x = a_1$, $a_y = a_2$, and $a_z = a_3$. Also, the usual summation convention has been used in connection with indicial notation. According to this convention, whenever the same letter subscript occurs twice in a term, that subscript is to be given all possible values and the results are to be added together. The symbol $\Sigma$ may then be omitted because the repeated subscript indicates the summation. For example, in three dimensions,

$$a_{i2}b_i = a_{12}b_1 + a_{22}b_2 + a_{32}b_3$$

The Latin indices (i, j, k, l, etc.) assume the values 1, 2, and 3, while the Greek indices ($\alpha$, $\beta$, $\gamma$, $\delta$, etc.) assume the values 1 and 2 only. Thus, these two sets of indices can be used for three-dimensional quantities and two-dimensional quantities respectively. In addition, the usual notation $A_{i,j}$ which denotes $\frac{\partial A_i}{\partial X_j}$ has been used throughout this dissertation.

In the present analysis, the Modified Hellinger-Reissner Principle has been taken as the starting point from which subsequent analyses leading to various special cases have emerged in a number of steps. The Hellinger-Reissner Principle and the Principle of Minimum Potential Energy can be interpreted as special cases of the Generalized Principle of Minimum Potential Energy [48]. The Hellinger-Reissner Principle as given in [48] involves linear strain-displacement relations, while the Modified Hellinger-Reissner Principle, used by Habip [41] on the basis of [49] is also applicable to non-linear strain displacement relations.

## Three-Dimensional Analysis

### Modified Hellinger-Reissner Principle

If for an elastic body

${}_oV$, $dV$ = Volume of the undeformed body, related element of volume

${}_{\circ}A_{s(t)}$, $dA$ = Total surface area of the undeformed body over which stresses are prescribed, related element of area

${}_{\circ}A_{V(t)}$, $dA$ = Total surface area of the undeformed body over which displacements are prescribed, related element of area

$V_i$ = Components of the displacement vector referred to the base vectors of the undeformed body

$\gamma_{ij}$ = Strain tensor

$S_{ij}$ = 2nd Piola-Kirchhoff stress tensor as defined in [50]

$\rho_0$ = Density of the undeformed body

${}_{\circ}f_i$ = Components of acceleration vector referred to the base vectors of the undeformed body

${}_{\circ}F_i$ = Components of the body forces per unit mass of the undeformed body referred to the base vectors of the undeformed body

$\varepsilon^*$ = Strain energy function per unit volume of the undeformed body

$S_{i*}$ = Components of the stress vector ${}_{\circ}S$ referred to the base vectors in the undeformed body

$$\delta V \quad = \text{Variation of a quantity } V \tag{1}$$

And if tildes (∿) indicate prescribed quantities, then the Modified Hellinger-Reissner Principle as given in [41] asserts that the variational principle

$$\int_{{}^{\circ}V} \rho_\circ \left({}_\circ f_i - {}_\circ F_i\right) \delta V_i \, dV = \delta \Bigg[ \int_{{}^{\circ}V} \left(S_{ij}\gamma_{ij} - \varepsilon^*\right) dV - \int_{{}^{\circ}V} \tfrac{1}{2} S_{ij}\left(V_{i,j} + V_{j,i} + V_{k,i}V_{k,j}\right) dV$$
$$+ \int_{{}_\circ A_{s(t)}} \tilde{S}_{i*} V_i \, dA + \int_{{}_\circ A_{V(t)}} S_{i*}\left(V_i - \tilde{V}_i\right) da \Bigg] \tag{2}$$

where $\gamma_{ij}$, $S_{ij}$, $V_i$ and $S_{i*}$ are varied independently, is equivalent to Cauchy's first law in ${}_\circ V$, to the stress boundary condition on the part ${}_\circ A_{s(t)}$ of the boundary, to the displacement boundary condition on the remaining part ${}_\circ A_V$, and to the stress-strain and strain-displacement relations in ${}_\circ V$, when the symmetries of $\gamma_{ij}$ and $S_{ij}$ are both used.

Habip [41] has demonstrated the proof of the above stated theorem. In this dissertation, the following strain displacement relations

$$\gamma_{\alpha\beta} = \tfrac{1}{2}\left(V_{\alpha,\beta} + V_{\beta,\alpha} + V_{3,\alpha}V_{3,\beta}\right)$$
$$\gamma_{\alpha 3} = \tfrac{1}{2}\left(V_{\alpha,3} + V_{3,\alpha}\right)$$

$$\gamma_{33} = V_{3,3} \tag{3}$$

have been used instead of the exact relations

$$\gamma_{ij} = \tfrac{1}{2}\left(V_{i,j} + V_{j,i} + V_{k,i}V_{k,j}\right) \tag{4}$$

Relations (3) have been used by many authors, e.g., [19], [51], and are realistic considering the degree of non-linearity and therefore complexity involved for the corresponding gain in accuracy. However, for very large deformations, Equation (4) must be used.

Corresponding to the strain-displacement relations adopted here and given by (3), the Modified Hellinger-Reissner Principle given by (2) changes to

$$\int_{{}_\circ V} \rho_\circ \left({}_\circ f_\alpha - {}_\circ F_\alpha\right) \delta V_\alpha dV + \int_{{}_\circ V} \rho_\circ \left({}_\circ f_3 - {}_\circ F_3\right) \delta V_3 dV$$

$$= \delta\Bigg[\int_{{}_\circ V} \left(S_{\alpha\beta}\gamma_{\alpha\beta} + 2S_{\alpha3}\gamma_{\alpha3} + S_{33}\gamma_{33} - \varepsilon^*\right) dV$$

$$- \int_{{}_\circ V} \tfrac{1}{2} S_{\alpha\beta}\left(V_{\alpha,\beta} + V_{\beta,\alpha} + V_{3,\alpha}V_{3,\beta}\right) dV - \int_{{}_\circ V} S_{\alpha3}\left(V_{\alpha,3} + V_{3,\alpha}\right) dv - \int_{{}_\circ V} S_{33}V_{3,3} dV$$

$$+ \int_{{}_\circ A_{s(t)}} \left(\tilde{S}_{\alpha*}V_\alpha + \tilde{S}_{3*}V_3\right) dA + \int_{{}_\circ A_{v(t)}} S_{\alpha*}\left(V_\alpha - \tilde{V}_\alpha\right) dA + \int_{{}_\circ Av(t)} S_{3*}\left(V_3 - \tilde{V}_3\right) dA\Bigg] \tag{5}$$

Equation (5) is the governing equation of the present analysis. In both sides of Equation (5), the operations of variation ($\delta$) are carried out within the integrals and Green's theorem is made use of to convert volume integrals into surface integrals. To avoid writing down very lengthy expressions unnecessarily, only a sample calculation for one of the expressions in Equation (4) is shown below, calculations for the other expressions being quite similar.

$$\int_{_oV} S_{\alpha\beta}\delta V_{\alpha,\beta} = \int_{_oV} \left(S_{\alpha\beta}\delta V_{\alpha}\right)_{,\beta} dV - \int_{_oV} \left(S_{\alpha\beta}\right)_{,\beta}\delta V_{\alpha} dV$$

$$= \int_{_oV} \left(S_{\alpha i}\delta V_{\alpha}\right)_{,i} dV - \int_{_oV} \left(S_{\alpha 3}\delta V_{\alpha}\right)_{,3} dV - \int_{_oV} \left(S_{\alpha\beta}\right)_{,\beta}\delta V_{\alpha} dV$$

$$= \int_{_oA(t)} S_{\alpha i}\delta V_{\alpha}\,{}_on_i\, dA - \int_{_oV} \left(S_{\alpha 3}\delta V_{\alpha}\right)_{,3} dv - \int_{_oV} \left(S_{\alpha\beta}\right)_{,\beta}\delta V_{\alpha} dV$$

$$= \int_{_oA(t)} S_{\alpha\beta}\delta V_{\alpha}\,{}_on_{\beta}\, dA + \int_{_oA(t)} S_{\alpha 3}\,\delta V_{\alpha}\,{}_on_3\, dA - \int_{_oV} \left(S_{\alpha 3}\delta V_{\alpha}\right)_{,3} dv - \int_{_oV} \left(S_{\alpha\beta}\right)_{,\beta}\delta V_{\alpha} dV$$

$$= \int_{_oA_{s(t)}} S_{\alpha\beta}\,{}_on_{\beta}\,\delta V_{\alpha} dA + \int_{_oA_{v(t)}} S_{\alpha\beta}\,{}_on_{\beta}\,\delta V_{\alpha} dA + \int_{_oA_{s(t)}} S_{\alpha 3}\,{}_on_3\,\delta V_{\alpha} dA + \int_{_oA_{v(t)}} S_{\alpha 3}\,{}_on_3\,\delta V_{\alpha} dA$$

(contd.)

$$- \int_{{}_oV} (S_{\alpha 3} \delta V_{\alpha})_{,3} \, dV - \int_{{}_oV} (S_{\alpha\beta})_{,\beta} \, \delta V_{\alpha} \, dV \qquad (6)$$

where

${}_oA_{(t)}$ = Total surface of the body

${}_on_i$ = Cosine of the angle between the outward normal to the surface of the body at the point concerned and the coordinate axis $x_i$ referred to the undeformed body. (7)

After carrying out the calculations for all the expressions in Equation (5) in a way similar to that shown in (6) and collecting all the expressions on one side, Equation (5) takes the following form:

$$(A) + (B) + (C) = 0 \qquad (8)$$

where

$$(A) = \int_{{}_oV} \left[\gamma_{\alpha\beta} - \tfrac{1}{2}(V_{\alpha,\beta} + V_{\beta,\alpha} + V_{3,\alpha} V_{3,\beta})\right] \delta S_{\alpha\beta} dV + \int_{{}_oV} 2\left[\gamma_{\alpha 3} - \tfrac{1}{2}(V_{\alpha,3} + V_{3,\alpha})\right] \delta S_{\alpha 3} dV$$

$$+ \int_{{}_oV} (\gamma_{33} - V_{3,3}) \delta S_{33} dV + \int_{{}_oV} \left(S_{\alpha\beta} - \frac{\partial \varepsilon^*}{\partial \gamma_{\alpha\beta}}\right) \gamma\delta_{\alpha\beta} dV + \int_{{}_oV} 2\left(S_{\alpha 3} - \frac{\partial \varepsilon^*}{\partial \gamma_{\alpha 3}}\right) \delta\gamma_{\alpha 3} dV$$

$$+ \int_{{}_oV} \left(S_{33} - \frac{\partial \varepsilon^*}{\partial \gamma_{33}}\right) \delta\gamma_{33} dV$$

$$(B) = \int_{{}_{\circ}A_{s(t)}} (\tilde{S}_{\alpha *} - S_{\alpha\beta}{}_{\circ}n_{\beta} - S_{\alpha 3}{}_{\circ}n_{3})\, \delta V_{\alpha} dA$$

$$+ \int_{{}_{\circ}A_{s(t)}} \left\{ \tilde{S}_{3*} - (S_{\alpha\beta}V_{3,\beta} + S_{\alpha 3})_{\circ}n_{\alpha} - (S_{3\beta}V_{3,\beta} + S_{33})_{\circ}n_{3} \right\} \delta V_{3} dA$$

$$+ \int_{{}_{\circ}A_{v(t)}} (S_{\alpha *} - S_{\alpha\beta}{}_{\circ}n_{\beta} - S_{\alpha 3}{}_{\circ}n_{3}) \delta V_{\alpha} dA + \int_{{}_{\circ}A_{v(t)}} \left\{ S_{3*} - (S_{\alpha\beta}V_{3,\beta} + S_{\alpha 3})_{\circ}n_{\alpha} - (S_{3\beta}V_{3,\beta} + S_{33})_{\circ}n_{3} \right\} \delta V_{3} dA$$

$$+ \int_{{}_{\circ}V} \left\{ (S_{\alpha\beta})_{,\beta} + (S_{\alpha 3})_{,3} + \delta_{\circ}({}_{\circ}F_{\alpha} - {}_{\circ}f_{\alpha}) \right\} \delta V_{\alpha} dV + \int_{{}_{\circ}V} \left\{ (S_{\alpha\beta}V_{3,\beta} + S_{\alpha 3})_{,\alpha} + (S_{33})_{,3} + \right.$$

$$\left. \rho_{\circ}({}_{\circ}F_{3} - {}_{\circ}f_{3}) \right\} \delta V_{3} dV + \int (S_{3\beta}V_{3,\beta}\delta V_{3})_{,3}\, dV$$

and

$$(C) = \int_{{}_{\circ}A_{v(t)}} (V_{\alpha} - \tilde{V}_{\alpha})\, \delta S_{\alpha *} dA + \int_{{}_{\circ}A_{v(t)}} (V_{3} - \tilde{V}_{3})\, \delta S_{3*} \qquad (9)$$

Up to this point, this analysis has been very general and three-dimensional with no reference to a plate-like two-dimensional problem. Equation (8) with the aid of (9) can be applied to any three-dimensional elasticity problem. To do so, one would have to use the fundamental lemma of the Calculus of Variations [52] and set each individual integrand contained in Equation (8) equal to zero. A complete set of equilibrium equations, elastic equations, compatibility relations and boundary conditions would be obtained. For two-dimensional problems, this has been done in the subsequent chapters, and the procedure for the three-dimensional problem is similar to the two-dimensional one. Since our aim is to analyze plate problems, from this point on, our analysis would be two-dimensional.

## Two-Dimensional Analysis

Figure (1) shows a body having two parallel and flat surfaces, and surrounded by a surface perpendicular to the two flat faces. The thickness is small compared to other dimensions of the body, so we can consider it to be a plate. The plate can have many layers or just a single layer.

The coordinate axes $x_1$, $x_2$, and $x_3$ are taken such that the $x_1x_2$ plane coincides with the middle plane of the plate.

Various quantities are defined as follows:

(A subscript "$_o$" refers to the undeformed state.)

${}_oV$ = Total volume of the plate

${}_oA$ = Total area of the middle surface

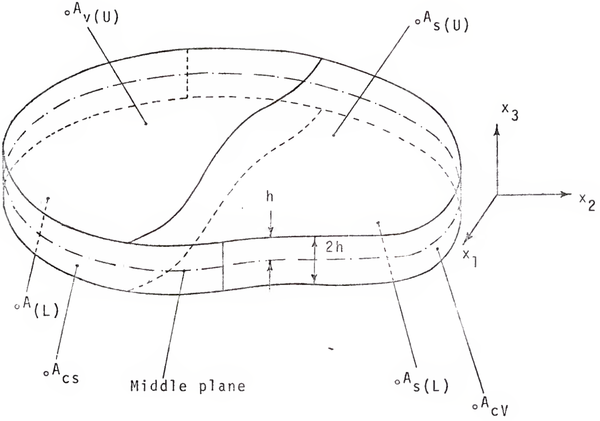


Figure 1. Plate with arbitrarily specified loads and displacements over its surface

${}_{\circ}A_{(t)}$ = Total surface of the plate

= ${}_{\circ}A_{s(t)} + {}_{\circ}A_{V(t)}$

where

${}_{\circ}A_{s(t)}$ and ${}_{\circ}A_{v(t)}$ have been defined in (1)

${}_{\circ}A_{s(t)} = {}_{\circ}A_{s(U+L)} + {}_{\circ}A_{cs}$

${}_{\circ}A_{s(U+L)}$ = Total portions of the upper and lower surfaces of the plate over which stresses are prescribed.

= ${}_{\circ}A_{s(U)} + {}_{\circ}A_{s(L)}$

${}_{\circ}A_{s(U)}$ = Portion of the upper surface over which stresses are prescribed

${}_{\circ}A_{s(L)}$ = Portion of the lower surface over which stresses are prescribed

${}_{\circ}A_{cs}$ = Portion of the edge surface of the plate over which stresses are prescribed

${}_{\circ}A_{V(t)} = {}_{\circ}A_{V(U+L)} + {}_{\circ}A_{cV}$

${}_{\circ}A_{V(U+L)}$ = Total portions of the upper and lower surfaces over which displacements are prescribed

= ${}_{\circ}A_{V(U)} + {}_{\circ}A_{V(L)}$

${}_{\circ}A_{V(U)}$ = Portion of the upper surface over which displacements are prescribed

${}_{\circ}A_{V(L)}$ = Portion of the Lower surface over which displacements are prescribed

$_{\circ}A_{cV}$ = Portion of the edge-surface over which displacements are prescribed.

In our case,

$$_{\circ}A_{s(U)} = {}_{\circ}A_{s(L)} \quad \text{in magnitude}$$
$$= {}_{\circ}A_{s} \quad \text{(say)}$$

Also,

$$_{\circ}A_{V(U)} = {}_{\circ}A_{V(L)} \quad \text{in magnitude}$$
$$= {}_{\circ}A_{V} \quad \text{(say)} \qquad (10)$$

Now the results of Equation (8) can be applied to the plate considered above. Due to difficulties in carrying out very lengthy calculations for different expressions all through simultaneously, the calculations resulting from the expressions (A), (B), and (C) of Equation (8) would be carried out separately. Expression (B) would be considered first.

In the expression for (B) given in Equation (9), each of the three volume integrals is converted into a double integral consisting of an area integral over the middle surface of the plate and a simple integral through the whole thickness of the plate. The surface integrals like $_{\circ}A_{s(t)}$ are broken up into two parts, $_{\circ}A_{s(U+L)}$ and $_{\circ}A_{cs}$. The surface integrals associated with $_{\circ}A_{cs}$ are then converted into a double integral, one of which is a line integral along ${}_{\circ}C_s$, the edge of the middle plane over which stress resultants are given, and the other one is a simple integral through the whole thickness of the plate. Then, noting that

$${}_{\circ}n_{\alpha} = 0 \text{ on the upper and lower surfaces}$$

and

$${}_{\circ}n_{3} = 0 \text{ on the edge,}$$

(B) can be expressed as follows:

$$(B) = (D) + (E) + (F) + (G) \tag{11}$$

where

$$(D) = \int_{{}_{\circ}A_{s(u+L)}} (\tilde{S}_{\alpha *} - S_{\alpha 3}\,{}_{\circ}n_{3})\delta V_{\alpha}dA + \int_{{}_{\circ}A_{s(U+L)}} \left\{ \tilde{S}_{3*} - (S_{3\beta}V_{3,\beta} + S_{33})\,{}_{\circ}n_{3} \right\} \delta V_{3}dA$$

$$+ \iiint_{{}_{\circ}A_{s}\,-h}^{h} \left\{ S_{\alpha\beta,\beta} + S_{\alpha 3,3} + \rho_{\circ}({}_{\circ}F_{\alpha} - {}_{\circ}f_{\alpha}) \right\} \delta V_{\alpha}dx_{3}dA$$

$$+ \iiint_{{}_{\circ}A_{s}\,-h}^{h} \left\{ (S_{\alpha\beta}V_{3,\beta} + S_{\alpha 3})_{,\alpha} + S_{33,3} + \rho_{\circ}({}_{\circ}F_{3} - {}_{\circ}f_{3}) \right\} \delta V_{3}dx_{3}dA +$$

$$+ \iiint_{{}_{\circ}A_{s}\,-h}^{h} \left\{ (S_{3\beta}V_{3,\beta}\delta V_{3})_{,3}\; dx_{3}dA \right\}$$

$$(E) = \int_{{}_{\circ}C_s}\int_{-h}^{h} (\tilde{S}_{\alpha *} - S_{\alpha\beta}\,{}_{\circ}n_{\beta})\,\delta V_{\alpha}\,dx_3\,ds + \int_{{}_{\circ}C_s}\int_{-h}^{h} \left\{\tilde{S}_{3*} - (S_{\alpha\beta}V_{3,\beta} + S_{\alpha 3})\,{}_{\circ}n_{\alpha}\right\}\delta V_3\,dx_3\,ds$$

$$(F) = \int_{{}_{\circ}A_{V(u+L)}} (S_{\alpha *} - S_{\alpha 3}\,{}_{\circ}n_3)\,\delta V_{\alpha}\,dA + \int_{{}_{\circ}A_V(U+L)} \left\{S_{3*} - (S_{3\beta}V_{3,\beta} + S_{33})\,{}_{\circ}n_3\right\}\,\delta V_3\,dA$$

$$+ \int_{{}_{\circ}A_V}\int_{-h}^{h} \left\{S_{\alpha\beta,\beta} + S_{\alpha 3,3} + \rho_{\circ}({}_{\circ}F_{\alpha} - {}_{\circ}f_{\alpha})\right\}\,\delta V_{\alpha}\,dx_3\,dA$$

$$+ \int_{{}_{\circ}A_V}\int_{-h}^{h} \left\{(S_{\alpha\beta}V_{3,\beta} + S_{\alpha 3})_{,\alpha} + S_{33,3} + \rho_{\circ}({}_{\circ}F_3 - {}_{\circ}f_3)\right\}\,\delta V_3\,dx_3\,dA \;+$$

$$\int_{{}_{\circ}A_V}\int_{-h}^{h} (S_{3\beta}V_{3,\beta}\,\delta V_3)_{,3}\,dx_3\,dA$$

and

$$(G) = \int_{{}_{\circ}C_s}\int_{-h}^{h} (S_{\alpha *} - S_{\alpha\beta}\,{}_{\circ}n_{\beta})\,\delta V_{\alpha}\,dx_3\,ds + \int_{{}_{\circ}C_V}\int_{-h}^{h} \left\{S_{3*} - S_{\alpha\beta}V_{3,\beta} + S_{\alpha 3})\,{}_{\circ}n_{\alpha}\right\}\delta V_3\,dx_3\,ds$$

(ds = Element of length along ${}_{\circ}C_s$ or ${}_{\circ}C_V$) (12)

The results obtained so far are applicable to plates consisting of any number of layers having different properties and geometry. But from here on, the analysis would depend on the type of plate considered viz. single layer or multiplayer plates, thin or moderately thick plates, etc., and also on the type of displacement functions $V_\alpha$ and $V_3$. Habip [41] has demonstrated the applicability of these results to a single layer plate.

In this dissertation, a three-layered plate, popularly known as a sandwich plate will be considered.

## Sandwich Plate

A sandwich plate consisting of three layers is shown in Figure (2). The face layers are much thinner than the core. All layers are uniformly thick throughout. The two faces are of the same thickness $t_f$, the core thickness being $t_c$.

The present analysis is capable of treating mixed boundary value problems. On the upper surface, the line AB separates the two regions ${}_{\circ}A_{V(u)}$ and ${}_{\circ}A_{s(U)}$ over which displacements and stresses respectively are prescribed. On the lower surface, the line CD separates the corresponding regions. The line CD is located exactly below the line AB. Also, at any two points located on the upper and lower surfaces of the plate and having the same $x_1$ and $x_2$

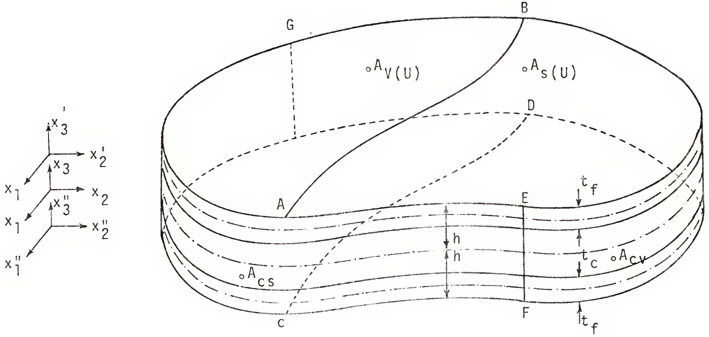


Figure 2. Sandwich plate of a general shape

coordinates, the same type of quantities (stresses or displacements) are specified. Therefore $_{\circ}A_{V(U)}$ is equal to $_{\circ}A_{V(L)}$ both in size and shape, and the former is located exactly above the latter, as shown in the figure. The same thing holds for $_{\circ}A_{s(U)}$ and $_{\circ}A_{s(L)}$. Finally, the lines EF and GH which separate the portions $_{\circ}A_{cs}$ and $_{\circ}A_{cV}$ of the edge surface of the plate, are straight lines perpendicular to the middle surface of the core. These conditions are necessary because they assure that the type of prescribed quantities (stresses or displacements) do not change across the thickness.

In addition to the conditions stated above, the following assumptions are made:

A1) The transverse normal strain in the face layers is negligible compared to that of the core.

A2) Displacements in each layer are linear functions of distances from the median plane of that layer.

A3) The core is attached to the faces securely.

For almost all the common shapes and material properties of a sandwich plate, assumption (A1) is valid. Typical geometry and material properties of commercially used sandwich plates can be found in [1] and [53]. Assumption (A2) has been used by [10], [11], [12] and many others. This is an approximation to a power series containing an infinite number of terms for the displacement functions. The validity of this assumption depends on the amount of deformation produced and the accuracy needed. However, retaining even a single nonlinear term in the expressions for the displacements give rise to inordinately cumbersome equations with no appreciable gain in accuracy. Thus this assumption is justified. Assumption (A3) rises out of the necessity of continuity of displacements at the interfaces of different layers.

Three sets of coordinates as shown in Figure (2) are chosen such that their origins lie on the same straight line perpendicular to the undeformed middle plane of the core. The planes containing $(x_1', x_2')$, $(x_1, x_2)$ and $(x_1'', x_2'')$ axes coincide with the middle planes of the upper layer, the core and the lower layer respectively, and the $x_3'$, $x_3$ and $x_3''$ axes are on the same straight line and in the same direction.

The single-primed and double-primed quantities refer to the upper and lower faces respectively. The unprimed quantities refer to the core. Then, the following relations hold:

$$x_3' = x_3 - \tfrac{1}{2}(t_c + t_f)$$

$$x_3'' = x_3 + \tfrac{1}{2}(t_c + t_f)$$

$$x'_\alpha = x_\alpha = x_\alpha'' \qquad (13)$$

Assumptions (A1) and (A2) lead to the following relations between different displacements:

$$
\begin{aligned}
V_\alpha' &= U'_\alpha\,(x_\beta) + x_3'\,\Psi_\alpha'(x_\beta) \\
V_3' &= U_3'(x_\beta) \\
V_\alpha'' &= U_\alpha''(x_\beta) + x_3''\Psi_\alpha''\,(x_\beta) \\
V_3'' &= U_3''(x_\beta) \\
V_\alpha &= U_\alpha(x_\beta) + x_3\Psi_\alpha\,(x_\beta) \\
V_3 &= U_3(x_\beta) + x_3\Psi_3(x_\beta)
\end{aligned}
\qquad (14)
$$

where $U_\alpha'$, $U_3'$, $\Psi_\alpha'$, $U_\alpha''$, $U_3''$, $\Psi_\alpha''$, $U_\alpha$, $U_3$, $\Psi_\alpha$ and $\Psi_3$ are called displacement functions and are functions of $x_1$ and $x_2$ only. There are 16 displacement functions in Equation (14). But the assumption (A3) gives rise to the following relations among these quantities:

$$
\begin{aligned}
V'_\alpha(x_1,\ x_2,\ x_3' = -\tfrac{tf}{2}) &= V_\alpha(x_1,\ x_2,\ x_3 = \tfrac{tc}{2}) \\
V'_3(x_1,\ x_2,\ x_3' = -\tfrac{tf}{2}) &= V_3(x_1,\ x_2,\ x_3 = \tfrac{tc}{2}) \\
V_\alpha''(x_1,\ x_2,\ x_3'' = \tfrac{tf}{2}) &= V_\alpha(x_1,\ x_2,\ x_3 = -\tfrac{tc}{2}) \\
V_3''(x_1,\ x_2,\ x_3'' = \tfrac{tf}{2}) &= V_3(x_1,\ x_2,\ x_3 = -\tfrac{tc}{2})
\end{aligned}
\qquad (15)
$$

There are six equations in (15). Thus the number of independent displacement functions is reduced from 16 to 10 by combining (14) and (15). Application of conditions (15) to (14) gives

$$\Psi'_\alpha = \frac{2}{t_f}(U'_\alpha - U_\alpha - \frac{t_c}{2}\Psi_\alpha)$$

$$U_3' = U_3 + \frac{t_c}{2}\Psi_3$$

$$\Psi_\alpha'' = \frac{2}{t_f}(U_\alpha - U_\alpha'' - \frac{t_c}{2}\Psi_\alpha)$$

$$U_3'' = U_3 - \frac{t_c}{2}\Psi_3 \qquad (16)$$

Substitution of (13) and (16) into (14) gives the expressions for the displacements in terms of the displacement functions as follows:

$$V_\alpha' = U'_\alpha\,[1 + \frac{2}{t_f}\{x_3 - \tfrac{1}{2}(t_c + t_f)\}\,] - U_\alpha\frac{2}{t_f}\{x_3 - \tfrac{1}{2}(t_c + t_f)\} - \Psi_\alpha\frac{t_c}{t_f}\{x_3 - \tfrac{1}{2}(t_c + t_f)\}$$

$$V_3' = U_3 + \frac{t_c}{2}\Psi_3$$

$$V_\alpha'' = U''_\alpha[1 - \frac{2}{t_f}\{x_3 + \tfrac{1}{2}(t_c + t_f)\}\,] + U_\alpha\frac{2}{t_f}\{x_3 + \tfrac{1}{2}(t_c + t_f)\} - \Psi_\alpha\frac{t_c}{t_f}\{x_3 + \tfrac{1}{2}(t_c + t_f)\}$$

$$V_3'' = U_3 - \frac{t_c}{2}\Psi_3$$

$$V_\alpha = U_\alpha + x_3\Psi_\alpha$$

$$V_3 = U_3 + x_3\Psi_3 \qquad (17)$$

The expressions given by (17) are then substituted into the expressions (D), (E), (F), and (G) given by (12) and in the expression (C) given by (9). The integrations are then carried out through the thickness of the plate.

The expressions that emerge from each of the quantities (D), (E), (F), etc. are excessively long in the intermediate steps making it extremely inconvenient to carry out calculations of all of them simultaneously. Therefore, calculations would be shown for (D) only, those for the rest being similar.

But the calculations for each of these quantities, like (D) alone, involve very long expressions in the intermediate steps, and require a great amount of writing space to show the procedure followed. On the other hand, omitting all the intermediate steps may make it very hard to know how the final results were determined. Therefore only those intermediate steps which are essential to keep track of the procedure followed will be shown.

Noting that

$$
{}_{\circ}n_3 = 1 \quad \text{on the upper surface}
$$

$$
= -1 \quad \text{on the lower surface}
$$

and using definitions (10), the first integral of the expression (D) is given as:

$$
\int_{{}_{\circ}A_{s(U+L)}} (\tilde{S}_{\alpha *} - S_{\alpha 3}\,{}_{\circ}n_3)\,\delta V_\alpha\, dA = \int_{{}_{\circ}A_s} \left\{ (S'_{\alpha *})_h - (S'_{\alpha 3})_h \right\} \delta V'_\alpha\, dA + \int_{{}_{\circ}A_s} \left\{ (S''_{\alpha *})_{-h} + (S''_{\alpha 3})_{-h} \right\} \delta V''_\alpha\, dA
$$

$$= \int_{\circ A_s} \{2(\tilde{S}'_{\alpha *})_h - 2(S'_{\alpha 3})_h\} \delta U'_\alpha \, dA + \int_{\circ A_s} \left\{2(\tilde{S}''_{\alpha *})_{-h} + 2(S''_{\alpha 3})_{-h}\right\} \delta U''_\alpha \, dA$$

$$- \int_{\circ A_s} \{(\tilde{S}''_{\alpha *})_{-h} + (S''_{\alpha 3})_{-h} + (\tilde{S}'_{\alpha *})_h - (S'_{\alpha 3})_h\} \delta U_\alpha dA$$
$$+ \int_{\circ A_s} \{\frac{t_c}{2} (\tilde{S}''_{\alpha *})_{-h} - \frac{t_c}{2} (\tilde{S}'_{\alpha *})_h + \frac{t_c}{2} (S''_{\alpha 3})_{-h} + \frac{t_c}{2} (S'_{\alpha 3})_h \} \delta \Psi_\alpha dA \qquad (18)$$

where the following notation has been used:

$(x)_y$ = value of x at $x_3 = y$.

In a way similar to the one just shown, the second integral in the expression of D is

$$\int_{\circ A_{s(U+L)}} \{\tilde{S}_{3*} - (S_{3\beta} V_{3,\beta} + S_{33})_\circ n_3\} \delta V_3 dA = \int_{{}^\circ A_s} \{(\tilde{S}''_{3*})_{-h} + (\tilde{S}'_{3*}) + (S''_{3\beta})_{-h} U_{3,\beta}$$

$$- \frac{t_c}{2} (S''_{3\beta})_{-h} \Psi_{3,\beta} + (S''_{33})_{-h} - (S'_{3\beta})_h U_{3,\beta} - \frac{t_c}{2} (S'_{3\beta})_h \Psi_{3,\beta}$$

$$- (S'_{33})_h\} \delta U_3 dA$$

$$+ \int_{\circ A_s} \{ - \frac{tc}{2} (\tilde{S}''_{3*})_{-h} + \frac{t_c}{2} (\tilde{S}'_{3*})_h - \frac{t_c}{2} (S''_{3\beta})_{-h} U_{3,\beta} + \frac{t_c{}^2}{4} (S''_{3\beta})_{-h} \Psi_{3,\beta}$$

$$- \frac{t_c}{2} (S''_{33})_{-h} - \frac{t_c}{2} (S'_{3\beta}) U_{3,\beta} - \frac{t_c{}^2}{4} (S'_{3\beta})_h \Psi_{3,\beta} - \frac{t_c}{2} (S'_{33})_h\} \delta \Psi_3 dA \qquad (19)$$

The third integral in the expression for (D) is considered next. The integral through the whole thickness of the composite plate between the limits -h and +h is divided into three integrals, the first one between the limits -h and $-\frac{t_c}{2}$, the second one between the limits $-\frac{t_c}{2}$ and $\frac{t_c}{2}$, and the last one between the limits $\frac{t_c}{2}$ and h, corresponding to the lower face, the core and the upper face respectively. For each layer, the appropriate expressions for displacements are substituted in the integral for that particular layer. Thus,

$$\iint_{{}_oA_s}\int_{-h}^{h}\{S_{\alpha\beta,\beta} + S_{\alpha 3,3} + \rho_o({}_oF_\alpha - {}_of_\alpha)\}\delta V_\alpha dx_3 dA$$

$$= \iint_{{}_oA_s}\int_{-h}^{-\frac{t_c}{2}}\Big\{S''_{\alpha\beta,\beta} + \rho_o''({}_oF_\alpha'' - {}_of_\alpha'')\Big\}\delta V''_\alpha\, dx_3 dA + \iint_{{}_oA_s}\int_{-\frac{t_c}{2}}^{\frac{t_c}{2}}\Big\{S_{\alpha\beta,\beta} + \rho_o({}_oF_\alpha - {}_of_\alpha)\Big\}\delta V_\alpha dx_3 dA$$

$$+ \iint_{{}_oA_s}\int_{\frac{t_c}{2}}^{h}\{S'_{\alpha\beta,\beta} + \rho_o'({}_oF'_\alpha - {}_of'_\alpha)\}\delta V'_\alpha\, dx_3 dA + \iint_{{}_oA_s}\int_{-h}^{-\frac{t_c}{2}} S''_{\alpha 3,3}\, V''_\alpha\, dx_3 dA$$

$$+ \iint_{{}_oA_s}\int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} S_{\alpha 3,3}\delta V_\alpha dx_3 dA + \iint_{{}_oA_s}\int_{\frac{t_c}{2}}^{h} S'_{\alpha 3,3}\delta V_\alpha dx_3 dA \tag{20}$$

One important point regarding notation is to be noted in the above equation. The left hand side is a general expression which is applicable to a single layer or a multilayer plate. So all quantities in the left hand side are unprimed. But the right hand side is for a three-layered sandwich plate, so the unprimed quantities in the right hand side refer to the core, as stated earlier.

Now, the following changes of variables and limits of integration can be made:

$$\int_{-h}^{-\frac{t_c}{2}} Y(x_\alpha)\, dx_3 = \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} Y(x_\alpha)\, dx_3''$$

$$\int_{\frac{t_c}{2}}^{h} Y(x_\alpha)\, dx_3 = \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} Y(x_\alpha)\, dx_3' \qquad (21)$$

[using Equation (13)]

Then, substituting Equation (17) into (20) and using relations (21), the following results are obtained:

[The 3rd integral in the expression of (D)]

$$= \iint\limits_{{}_\circ A_s} \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} \{S''_{\alpha\beta,\beta} + \rho_\circ''({}_\circ F''_\alpha - {}_\circ f''_\alpha)\} \{\delta U''_\alpha (1 - \frac{2}{t_f} x_3'') + \delta U_\alpha \frac{2}{t_f} x_3'' - \delta \Psi_\alpha \frac{t_c}{t_f} x_3''\}\, dx_3'' dA$$

(Contd.)

$$+\iint_{{}_{\circ}A_s\;-\frac{t_c}{2}}^{\frac{t_c}{2}}\{S_{\alpha\beta,\beta}+\rho_{\circ}({}_{\circ}F_{\alpha}-{}_{\circ}f_{\alpha})\}\{(\delta U_{\alpha}+x_3\delta\Psi_{\alpha})\,dx_3dA$$

$$+\iint_{{}_{\circ}A_s\;-\frac{t_f}{2}}^{\frac{t_f}{2}}\{S'_{\alpha\beta,\beta}+\delta_{\circ}'({}_{\circ}F'_{\alpha}-{}_{\circ}f_{\alpha})\}\ \{\delta U'_{\alpha}(1+\frac{2}{t_f}x_3')-\delta U_{\alpha}\frac{2}{t_f}x_3'-\delta\Psi\frac{t_c}{t_f}x_3'\}\,dx_3'dA$$

$$+\iint_{{}_{\circ}A_2\;-h}^{-\frac{t_c}{2}}S''_{\alpha 3,3}\Big\langle\delta U''_{\alpha}[1-\frac{2}{t_f}\{x_3+\tfrac{1}{2}(t_c+t_f)\}]+\delta U_{\alpha}\frac{2}{t_f}\{x_3+\tfrac{1}{2}(t_c+t_f)\}$$

$$-\delta\Psi_{\alpha}\frac{t_c}{t_f}\{x_3+\tfrac{1}{2}(t_c+t_f)\}\Big\rangle dx_3dA+\iint_{{}_{\circ}A_x\;-\frac{t_c}{2}}^{\frac{t_c}{2}}S_{\alpha 3,3}(\delta U_{\alpha}+x_3\delta\Psi_{\alpha})dx_3dA$$

$$+\iint_{{}_{\circ}A_s\;\frac{t_c}{2}}^{h}S'_{\alpha 3,3}\Big\langle\delta U'_{\alpha}[1+\frac{2}{t_f}\{x_3-\tfrac{1}{2}(t_c+t_f)]\}-\delta U_{\alpha}\frac{2}{t_f}\{x_3-\tfrac{1}{2}(t_c+t_f)\}$$

$$-\delta\Psi_{\alpha}\frac{t_c}{t_f}\{x_3-\tfrac{1}{2}(t_c+t_f)\}\Big\rangle\ dx_3dA \qquad (22)$$

Similar procedure is applied to the fourth integral in the expression of (D). The last integral in the expression of (D) is now considered:

$$\iint\limits_{{}_{\circ}A_s\ -h}^{\quad h} (S_{3\beta}V_{3,\beta}\delta V_3)_{,}\ dx_3 dA$$

$$= \int\limits_{{}_{\circ}A_s} [(S'_{3\beta}V'_{3,\beta}\delta V'_3)_h - (S''_{3\beta}V''_{3,\beta}\delta V''_3)_{-h}]\ dA$$

$$= \int\limits_{{}_{\circ}A_s} [S'_{3\beta})_h\ U_{3,\beta} + \frac{t_c}{2}(S'_{3\beta})_h\ \Psi_{3,\beta} - (S''_{3\beta})\ U_{3,\beta} + \frac{t_c}{2}\ (S''_{3\beta})_{-h}\ \Psi_{3,\beta}]\ \delta U_3 dA$$

$$+ \int\limits_{{}_{\circ}A_s} [\frac{tc}{2}(S'_{3\beta})_h\ U_{3,\beta} + \frac{t_c2}{4}\ (S'_{3\beta})_h\ \Psi_{3,\beta} + \frac{t_c}{2}(S''_{3\beta})_{-h}\ U_{3,\beta} - \frac{t_c{}^2}{4}\ (S''_{3\beta})\Psi_{3,\beta}]\delta\Psi_3 dA \tag{23}$$

Then the final expressions given by (18), (19), (22), (23) and the one for the fourth integral in the expression of (D) are substituted into the first equation of (12). After that the remaining integrations are carried out and the terms within the same integral and associated with variation of the same quantity are grouped together. Then the terms like $-\frac{t_c}{t_f}(S'_{\alpha 3})_{\frac{t_c}{2}}$ and $\frac{t_c}{t_f}(S_{\alpha 3})_{\frac{t_c}{2}}$ are made to cancel each other because at the interface between the upper face and the core $(S'_{\alpha 3})_{\frac{t_c}{2}} = (S_{\alpha 3})_{\frac{t_c}{2}}$.

The next step in the present analysis is to define a number of quantities as follows:

$$(N'_{\alpha\beta},\ M'_{\alpha\beta},\ Q'_{\alpha},\ f'_{\alpha},\ F'_{\alpha},\ m'_{\alpha},\ M'_{\alpha}) = \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} (S'_{\alpha\beta},\ x_3{}'S_{\alpha\beta},\ S'_{\alpha 3}, \rho_o\ {}_of'_{\alpha},\ \rho_o\ {}_oF'_{\alpha},\ \rho_o x'_3\,{}_oF'_{\alpha})\ dx'_3$$

$$(N''_{\alpha\beta},\ M''_{\alpha\beta},\ Q''_{\alpha},\ f''_{\alpha},\ F''_{\alpha},\ m''_{\alpha},\ M''_{\alpha}) = \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} (S''_{\alpha\beta},\ x''_3 S_{\alpha\beta},\ S''_{\alpha 3},\ \rho_o\ {}_of''_{\alpha}, \rho_o\ {}_oF''_{\alpha},\ \rho_o x_3{}''\,{}_of''_{\alpha},\ \rho_o x_3{}''\,{}_oF''_{\alpha})dx_3{}''$$

and

$$(N_{\alpha\beta},\ M_{\alpha\beta},\ K_{\alpha\beta},\ Q_{\alpha},\ T_{\alpha},\ N_{33},\ f_{\alpha},\ F_{\alpha},\ m_{\alpha},\ M_{\alpha}) = \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} (S_{\alpha\beta}, x_3 S_{\alpha\beta}, x_3{}^2 S_{\alpha\beta},\ S_{\alpha 3},\ x_3 S_{\alpha 3},\ S_{33},\ \rho_o\ {}_of_{\alpha},\ \rho_o\ {}_oF_{\alpha},\ \rho_o x_3\,{}_of_{\alpha},\ \rho_o x_3\,{}_oF_{\alpha})\ dx_3 \qquad (24)$$

Using definitions (24), the final expression for the quantity (D) becomes:

$$(D) = \int_{{}_\circ A_s} \Big\langle [N'_{\alpha\beta,\beta} + \frac{2}{t_f} M'_{\alpha\beta,\beta} - \frac{2}{t_f} Q'_\alpha + F'_\alpha - f'_\alpha + \frac{2}{t_f} M'_\alpha - \frac{2}{t_f} m'_\alpha + 2(\tilde{S}'_{\alpha *})_h] \delta U'_\alpha$$

$$+ [N''_{\alpha\beta,\beta} - \frac{2}{t_f} M''_{\alpha\beta,\beta} + \frac{2}{t_f} Q''_\alpha + F''_\alpha - f''_\alpha - \frac{2}{t_f} M''_\alpha + \frac{2}{t_f} m''_\alpha + 2(\tilde{S}''_{\alpha *})_{-h}] \; \delta U''_\alpha$$

$$+ [\frac{2}{t_f} M''_{\alpha\beta,\beta} - \frac{2}{t_f} Q''_\alpha + N_{\alpha\beta,\beta} - \frac{2}{t_f} M'_{\alpha\beta,\beta} + \frac{2}{t_f} Q'_\alpha + \frac{2}{t_f} M''_\alpha - \frac{2}{t_f} m''_\alpha + F_\alpha - f_\alpha$$

$$- \frac{2}{t_f} M'_\alpha + \frac{2}{t_f} m'_\alpha - (\tilde{S}''_{\alpha *})_{-h} - (\tilde{S}'_{\alpha *})_h] \; \delta U_\alpha$$

$$+ [- \frac{t_c}{t_f} M''_{\alpha\beta,\beta} + \frac{t_c}{t_f} Q''_\alpha + M_{\alpha\beta,\beta} - Q_\alpha - \frac{t_c}{t_f} M'_{\alpha\beta,\beta} + \frac{t_c}{t_f} Q'_\alpha - \frac{t_c}{t_f} M''_\alpha + \frac{t_c}{t_f} m''_\alpha$$

$$+ M_\alpha - m_\alpha - \frac{t_c}{t_f} M'_\alpha + \frac{t_c}{t_f} m'_\alpha + \frac{t_c}{2} (\tilde{S}''_{\alpha *})_{-h} - \frac{t_c}{2} (\tilde{S}'_{\alpha *})_h] \; \delta \Psi_\alpha$$

$$+ [\{N''_{\alpha\beta} (U_{3,\beta} - \frac{t_c}{2} \Psi_{3,\beta}) + Q''_\alpha\}_{,\alpha} + (N_{\alpha\beta} U_{3,\beta} + M_{\alpha\beta} \Psi_{3,\beta} + Q_\alpha)_{,\alpha}$$

$$+ \{N'_{\alpha\beta} (U_{3,\beta} + \frac{t_c}{2} \Psi_{3,\beta}) + Q'_\alpha\}_{,\alpha} + F'_3 - f'_3 + F_3 - f_3 + F''_3 - f''_3 + (\tilde{S}''_{3*})_{-h} + (\tilde{S}'_{3*})_h] \; \delta U_3$$

$$+ [- \frac{t_c}{2} \{N''_{\alpha\beta} (U_{3,\beta} - \frac{t_c}{2} \Psi_{3,\beta}) + Q''_\alpha\}_{,\alpha} + (M_{\alpha\beta} U_{3,\beta} + K_{\alpha\beta} \Psi_{3,\beta} + T_\alpha)_{,\alpha} - N_{33}$$

$$+ \frac{t_c}{2} \{N'_{\alpha\beta} (U_{3,\beta} + \frac{t_c}{2} \Psi_{3,\beta}) + Q'_\alpha\}_{,\alpha} - \frac{t_c}{2} F''_3 + \frac{t_c}{2} f''_3 + M_3 - m_3$$

(Contd.)

$$+ \frac{t_c}{2} F'_3 - \frac{t_c}{2} f'_3 - \frac{t_c}{2} (\tilde{S}''_{3*})_{-h} + \frac{t_c}{2} (\tilde{S}'_{3*})_h ] \delta\Psi_3 \Big\rangle dA \tag{25}$$

The quantity (F) given in (12) is calculated in a way exactly similar to that for the quantity (D), and the resulting expression for (F) is exactly the same as for (D) given in (25) except that the quantities $\tilde{S}_{\alpha *}$ and $\tilde{S}_{3*}$ are to be replaced by $S_{\alpha *}$ and $S_{3*}$ respectively.

The procedure of evaluating the quantity (E) is similar to that for (D) just shown. Only the final result is given below:

$$(E) = \int_{{}_\circ C_s} \Big\langle [\tilde{S}'_\alpha + \frac{2}{t_f} \tilde{t}'_\alpha - (N'_{\alpha\beta} + \frac{2}{t_f} M'_{\alpha\beta}) {}_\circ n_\beta ] \delta U'_\alpha$$

$$+ [\tilde{S}''_\alpha - \frac{2}{t_f} \tilde{t}''_\alpha - (N''_{\alpha\beta} - \frac{2}{t_f} M''_{\alpha\beta}) {}_\circ n_\beta ] \delta U''_\alpha$$

$$+ [\frac{2}{t_f} \tilde{t}''_\alpha - \frac{2}{t_f} M''_{\alpha\beta} {}_\circ n_\beta + \tilde{S}_\alpha - N_{\alpha\beta} {}_\circ n_\beta - \frac{2}{t_f} \tilde{t}'_\alpha + \frac{2}{t_f} M'_{\alpha\beta} {}_\circ n_\beta ] \delta U_\alpha$$

$$+ [-\frac{\tilde{t}_c}{t_f} t''_\alpha + \frac{t_c}{t_f} M''_{\alpha\beta} {}_\circ n_\beta + \tilde{t}_\alpha - M_{\alpha\beta} {}_\circ n_\beta - \frac{t_c}{t_f} \tilde{t}'_\alpha + \frac{t_c}{t_f} M'_{\alpha\beta} {}_\circ n_\beta ] \delta\Psi_\alpha$$

$$+ [\tilde{S}''_3 + \tilde{S}_3 + \tilde{S}'_3 - \{N''_{\alpha\beta}(U_{3,\beta} - \frac{t_c}{2} \Psi_{3,\beta}) + Q''_\alpha\} {}_\circ n_\alpha - (N_{\alpha\beta} U_{3,\beta} + M_{\alpha\beta} \Psi_{3,\beta} + Q_\alpha) {}_\circ n_\alpha$$

$$- \{N'_{\alpha\beta}(U_{3,\beta} + \frac{t_c}{2} \Psi_{3,\beta}) + Q'_\alpha\} {}_\circ n_\alpha ] \delta U_3$$

$$+ [\frac{t_c}{2} \tilde{S}''_3 + \tilde{t}_3 + \frac{t_c}{2} \tilde{S}'_3 - \frac{t_c}{2} \{N''_{\alpha\beta}(U_{3,\beta} - \frac{t_c}{2} \Psi_{3,\beta}) + Q''_\alpha\} {}_\circ n_\alpha$$

(contd.)

$$-(M_{\alpha\beta}U_{3,\beta}+K_{\alpha\beta}\Psi_{3,\beta}+T_{\alpha})\circ n_{\alpha} - \frac{t_c}{2}\{N'_{\alpha\beta}(U_{3,\beta}+\frac{t_c}{2}\Psi_{3,\beta}) + Q'_{\alpha}\}\circ n_{\alpha}\Big\rangle dS \tag{26}$$

where, the following definitions have been used in addition to those given in (24):

$$(\tilde{S}'_{\alpha},\ \tilde{t}'_{\alpha},\ \tilde{S}'_3) = \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} (\tilde{S}'_{\alpha*},\ x'_3\,\tilde{S}_{\alpha*},\ \tilde{S}'_{3*})\ dx'_3$$

$$(\tilde{S}''_{\alpha},\ \tilde{t}''_{\alpha},\ \tilde{S}''_3) = \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} (\tilde{S}''_{\alpha*},\ x''_3\,\tilde{S}''_{\alpha*},\ \tilde{S}''_{3*})\ dx''_3$$

and

$$(\tilde{S}_{\alpha},\ \tilde{t}_{\alpha},\ \tilde{S}_3,\ \tilde{t}_3) = \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} (\tilde{S}_{\alpha*},\ x_3\,\tilde{S}_{\alpha*},\ \tilde{S}_{3*},\ x_3\,\tilde{S}_{3*})\ dx_3 \tag{27}$$

The expression (G) given in (12) is evaluated in a way which is similar to that for (E) just shown above. Also, the resulting expression for (G) is exactly the same as that for (E) given in (26) except that the tildes ($\sim$)

are removed from all quantities in which they occur, including the definitions (27).

The quantity (C) given in (9) can be evaluated by the procedure used already for the quantities (D), (E), etc., and the resulting expression for (C) is given below:

$$(C) = \int_{{}_\circ A_V} [V'_\alpha - \tilde{V}'_\alpha)\delta S'_{\alpha *} + (V''_\alpha - \tilde{V}''_\alpha)\delta S''_{\alpha *} + (V'_3 - \tilde{V}'_3)\delta S'_{3*} + (V''_3 - \tilde{V}''_3)\delta S''_{3*}]\, dA$$

$$+ \int_{{}_\circ C_V} \Big[ \int_{-h}^{-\frac{t_c}{2}} (V''_\alpha - \tilde{V}''_\alpha)\delta S''_{\alpha *} dx_3 + \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} (V_\alpha - \tilde{V}_\alpha)\delta S_{\alpha *} dx_3 + \int_{\frac{t_c}{2}}^{h} (V'_\alpha - \tilde{V}''_\alpha)\delta S'_{\alpha *}\, dx_3$$

$$+ \int_{-h}^{-\frac{t_c}{2}} (V''_3 - \tilde{V}''_3) dS''_{3*}\, dx_3 + \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} (V_3 - \tilde{V}_3)\delta S_{3*} dx_3 + \int_{\frac{t_c}{2}}^{h} (V'_3 - \tilde{V}'_3)\ \delta S'_{3*}\, dx_3 \Big] ds \qquad (28)$$

Finally, the quantity (A) given in (9) is now considered. Like the procedure followed previously, each single volume integral is first converted into a double integral, one of which is an area integral and the other one of which is an integral through the thickness. The procedure is demonstrated with the first integral in the expression of (A) given in (9).

$$\int_{{}_\circ V} [\gamma_{\alpha\beta} - \tfrac{1}{2}(V_{\alpha,\beta} + V_{\beta,\alpha} + V_{3,\alpha}V_{3,\beta})]\ \delta S_{\alpha\beta} dV$$

(contd.)

$$= \int_{{}_oA} \left\langle \int_{-h}^{-\frac{t_c}{2}} \{\gamma''_{\alpha\beta} - \tfrac{1}{2}(V''_{\alpha,\beta} + V''_{\beta,\alpha} + V''_{3,\alpha}V''_{3,\beta})\} dx_3 \delta S''_{\alpha\beta} \right.$$

$$+ \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \{\gamma_{\alpha\beta} - \tfrac{1}{2}(V_{\alpha,\beta} + V_{\beta,\alpha} + V_{3,\alpha}V_{3,\beta})\} dx_3 \delta S_{\alpha\beta} + \int_{\frac{t_c}{2}}^{h} \{\gamma'_{\alpha\beta} - (V'_{\alpha,\beta} + V'_{\beta,\alpha}$$

$$\left. + V'_{3,\alpha}V'_{3,\beta})\} \; dx'_3 \; \delta S'_{\alpha\beta} \right\rangle dA \qquad (29)$$

The following forms are assumed for the strains

$$\gamma_{\alpha\beta}(x_\alpha, x_3) = \sum_{n=0}^{M} x_3^{\,n} \; {}_n\gamma_{\alpha\beta} \; (x_\alpha)$$

$$\gamma'_{\alpha\beta}(x_\alpha, x') = \sum_{n=0}^{M'} x'^{\,n}_3 \; {}_n\gamma'_{\alpha\beta} \; (x_\alpha)$$

and

$$\gamma''_{\alpha\beta}(x_\alpha, x''_3) = \sum_{n=0}^{M''} x''^{\,n}_3 \; {}_n\gamma''_{\alpha\beta} \; (x_\alpha) \qquad (30)$$

where M, M' and M" are positive integers. The quantities ${}_n\gamma_{\alpha\beta}$, ${}_n\gamma'_{\alpha\beta}$ and ${}_n\gamma''_{\alpha\beta}$ are obtained in terms of the displacement functions given in (14) after (30) is substituted into (29) and into the second and the third integrals in the expression of (A).

The fourth, the fifth and the sixth integrals in the expression of (A) give the relationships between various stress resultants ($N_{\alpha\beta}$, $M_{\alpha\beta}$ etc.) and the strain functions (${}_n\gamma_{\alpha\beta}$) through the strain energy function per unit area ($\varepsilon$) for each layer. The procedure is demonstrated with the fourth integral.

$$\int_{{}_oV} (S_{\alpha\beta} - \frac{\partial\varepsilon^*}{\partial\gamma_{\alpha\beta}})\ \delta\gamma_{\alpha\beta} dV = \int_{{}_oA} \Big[ \int_{-h}^{-\frac{t_c}{2}} (S''_{\alpha\beta} - \frac{\partial\varepsilon^{*''}}{\partial\gamma''_{\alpha\beta}})\ \delta\gamma''_{\alpha\beta} dx_3$$

$$+ \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} (S_{\alpha\beta} - \frac{\partial\varepsilon^*}{\partial\gamma_{\alpha\beta}})\ \delta\gamma_{\alpha\beta} dx_3 + \int_{\frac{t_c}{2}}^{h} (S'_{\alpha\beta} - \frac{\partial\varepsilon^{*'}}{\partial\gamma'_{\alpha\beta}})\ \delta\gamma'_{\alpha\beta}\ dx_3 \Big]\ dA \qquad (31)$$

Now, $$\int_{-h}^{-\frac{t_c}{2}} (S''_{\alpha\beta} - \frac{\partial\varepsilon^{*''}}{\partial\gamma''_{\alpha\beta}})\ \delta\gamma''_{\alpha\beta}\ dx_3 = \int_{-\frac{t_f}{2}}^{-\frac{t_c}{2}} (S''_{\alpha\beta} - \frac{\partial\varepsilon^{*''}}{\partial\gamma''_{\alpha\beta}})\ \delta\gamma''_{\alpha\beta}\ dx_3'' \qquad (32)$$

[by applying (21)]

From (30), the following result is obtained:

$$\frac{\partial\varepsilon^{*''}}{\partial{}_o\gamma''_{\alpha\beta}} = \frac{\partial\varepsilon^{*''}}{\partial\gamma''_{\mu\nu}} \frac{\partial\gamma''_{\mu\nu}}{\partial{}_o\gamma''_{\alpha\beta}} = \frac{\partial\varepsilon^{*''}}{\partial\gamma''_{\mu\nu}} \frac{\partial}{\partial{}_o\gamma''_{\alpha\beta}} ({}_o\gamma''_{\mu\nu} + {}_1\gamma''_{\mu\nu})$$

$$= \frac{\partial\varepsilon^{*''}}{\partial\gamma''_{\alpha\beta}} \qquad (33)$$

Similarly,

$$\frac{\partial \varepsilon^{*''}}{\partial_1 \gamma''_{\alpha\beta}} = x_3'' \frac{\partial \varepsilon^{*''}}{\partial \gamma''_{\alpha\beta}} \tag{34}$$

Also,

$$\delta\gamma''_{\alpha\beta} = \delta_{\circ}\gamma''_{\alpha\beta} + \delta_1\gamma''_{\alpha\beta}\ x_3'' \tag{35}$$

With the aid of (33), (34) and (35), the right hand side of Equation (32) takes the form:

$$\int_{-h}^{-\frac{t_c}{2}} [(S''_{\alpha\beta} - \frac{\partial \varepsilon^{*''}}{\partial_{\circ}\gamma''_{\alpha\beta}})\ \delta_{\circ}\gamma''_{\alpha\beta} + (x_3''\ S''_{\alpha\beta} - \frac{\partial \varepsilon^{*''}}{\partial_1\gamma''_{\alpha\beta}})\ \delta_1\gamma''_{\alpha\beta}\ ]\ dx_3$$

or,

$$\int_{-h}^{-\frac{t_c}{2}} (S''_{\alpha\beta} - \frac{\partial \varepsilon^{*''}}{\partial \gamma''_{\alpha\beta}})\ \delta\gamma''_{\alpha\beta}\ dx_3 = (N''_{\alpha\beta} - \frac{\partial \varepsilon''}{\partial_{\circ}\gamma''_{\alpha\beta}})\ \delta_{\circ}\gamma''_{\alpha\beta} + (M''_{\alpha\beta} - \frac{\partial \varepsilon''}{\partial_1\gamma_{\alpha\beta}})\ \delta_1\gamma''_{\alpha\beta} \tag{36}$$

where

$$\varepsilon' = \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} \varepsilon^{*'}\ dx_3' \ ;\ \varepsilon'' = \int_{-\frac{t_f}{2}}^{\frac{t_f}{2}} \varepsilon^{*''}\ dx_3'' \ ;\ \varepsilon = \int_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \varepsilon^{*}\ dx_3 \tag{37}$$

The second and third integrals on the right hand side of Equation (31) are evaluated in a similar way. The procedure is then repeated for the fifth and the sixth integrals in the expression of (A) given in (9).

Now the resulting expressions of the quantities (D), (E), (F) and (G) are added to obtain the quantity (B), and then to the quantity (B) the values of the quantities (A) and (C) are added. This combined expression is equated to zero as per Equation (8). Since all the independent quantities can be varied independently of each other, then by the fundamental lemma of Calculus of Variations, each expression within a separate integral and associated with the variation of an independent quantity must be equal to zero. All the governing equations for a sandwich plate are then generated.

At this point it is to be noted that the original three-dimensional sandwich plate problem has now been reduced to a purely two-dimensional problem [Figure (3)].

Figure (3) shows a flat surface of total area ${}_oA$ and bounded by a closed curve ${}_oC$. The total area ${}_oA$ is divided into two parts, ${}_oA_V$ and ${}_oA_s$, by a line AB (which can be a closed curve, or a combination of closed curves in the general case), and over these two parts, displacements and stresses, respectively, are prescribed. The periphery ${}_oC$ is divided into two parts, DAC and CBD, which are

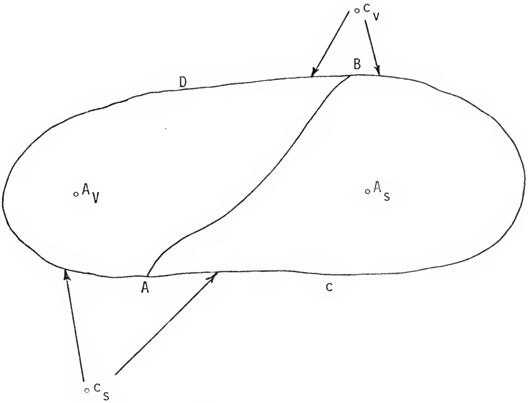


Figure 3. Two-dimensional mixed boundary value problem

called $_\circ C_s$ and $_\circ C_V$ respectively. Prescribed quantities are stresses over $_\circ C_s$ and displacements over $_\circ C_V$. It is therefore a mixed boundary value problem.

Equilibrium Equations Over $_\circ A_s$

$$N'_{\alpha\beta,\beta} + \frac{2}{t_f}(M'_{\alpha\beta,\beta} - Q'_\alpha) + F'_\alpha - f'_\alpha + \frac{2}{t_f}(M'_\alpha - m'_\alpha) + 2(\tilde{S}'_{\alpha*})_h = 0$$

$$N''_{\alpha\beta,\beta} - \frac{2}{t_f}(M''_{\alpha\beta,\beta} - Q''_\alpha) + F''_\alpha - f''_\alpha - \frac{2}{t_f}(M''_\alpha - m''_\alpha) + 2(\tilde{S}'_{\alpha*})_{-h} = 0$$

$$\frac{2}{t_f}(Q'_\alpha - M'_{\alpha\beta,\beta} - Q''_\alpha + M''_{\alpha\beta,\beta}) + N_{\alpha\beta,\beta} + F_\alpha - f_\alpha$$
$$- \frac{2}{t_f}(M'_\alpha - m'_\alpha - M''_\alpha + m''_\alpha) - (\tilde{S}'_{\alpha*})_h - (\tilde{S}''_{\alpha*})_{-h} = 0$$

$$\frac{t_c}{t_f}(Q'_\alpha - M'_{\alpha\beta,\beta} + Q''_\alpha - M''_{\alpha\beta,\beta}) + M_{\alpha\beta,\beta} - Q_\alpha + M_\alpha - m_\alpha$$

$$- \frac{t_c}{t_f}(M'_\alpha - m'_\alpha + M''_\alpha - m''_\alpha) - \frac{t_c}{2}\{(\tilde{S}'_{\alpha*})_h - (\tilde{S}''_{\alpha*})_{-h}\} = 0$$

$$\{N'_{\alpha\beta}(U_{3,\beta} + \frac{t_c}{2}\Psi_{3,\beta}) + N''_{\alpha\beta}(U_{3,\beta} - \frac{t_c}{2}\Psi_{3,\beta}) + N_{\alpha\beta}U_{3,\beta} + M_{\alpha\beta}\Psi_{3,\beta}$$

$$+ Q'_\alpha + Q''_\alpha + Q_\alpha\}_{,\alpha} + F'_3 - f'_3 + F''_3 - f''_3 + F_3 - f_3 + (\tilde{S}'_{3*})_h + (\tilde{S}''_{3*})_{-h} = 0$$

and

$$\frac{t_c}{2}\{N'_{\alpha\beta}(U_{3,\beta} + \frac{t_c}{2}\Psi_{3,\beta}) + Q'_\alpha\}_{,\alpha} - \frac{t_c}{2}N''_{\alpha\beta}(U_{3,\beta} - \frac{t_c}{2}\Psi_{3,\beta}) + Q''_\alpha\}_{,\alpha}$$

$$+ (M_{\alpha\beta}U_{3,\beta} + K_{\alpha\beta}\Psi_{3,\beta} + T_\alpha)_{,\alpha} - N_{33} + \frac{t_c}{2}(F'_3 - f'_3 - F''_3 + f''_3)$$

$$+ M_3 - m_3 + \frac{t_c}{2}(\tilde{S}'_{3*})_h - (\tilde{S}''_{3*})_{-h} = 0 \qquad (38)$$

Equilibrium equations over ${}_\circ A_V$ are exactly the same as those on ${}_\circ A_s$ except that the tildes ($\sim$) are removed from all the quantities in which they appear.

## Boundary Conditions

(1) On ${}_\circ C_s$

$$(\tilde{S}'_\alpha)_h - \frac{2}{t_f}(\tilde{t}'_\alpha)_h - (N'_{\alpha\beta} - \frac{2}{t_f}M'_{\alpha\beta})\,{}_\circ n_\beta = 0$$

$$(\tilde{S}''_\alpha)_{-h} + \frac{2}{t_f}(\tilde{t}''_\alpha)_{-h} - (N''_{\alpha\beta} + \frac{2}{t_f}M''_{\alpha\beta})\,{}_\circ n_\beta = 0$$

$$\frac{2}{t_f}\{(\tilde{t}'_\alpha)_h - (\tilde{t}''_\alpha)_{-h}\} + \tilde{S}_\alpha - \frac{2}{t_f}(M'_{\alpha\beta} - M''_{\alpha\beta})\ {}_\circ n_\beta - N_{\alpha\beta}\,{}_\circ n_\beta = 0$$

$$\frac{t_c}{t_f}\{(\tilde{t}'_\alpha)_h + (\tilde{t}''_\alpha)_{-h}\} - \tilde{t}_\alpha - \frac{t_c}{t_f}(M'_{\alpha\beta} + M''_{\alpha\beta})\ {}_\circ n_\beta + M_{\alpha\beta}\,{}_\circ n_\beta = 0$$

$$(\tilde{S}'_3)_h + \tilde{S}_3 + (\tilde{S}''_3)_{-h} - \{(N'_{\alpha\beta} + N_{\alpha\beta} + N''_{\alpha\beta})\ U_{3,\beta} - (\frac{t_c}{2} N'_{\alpha\beta} - \frac{t_c}{2} N''_{\alpha\beta}$$

$$+ M_{\alpha\beta})\ \Psi_{3,\beta} + Q'_\alpha + Q_\alpha + Q''_\alpha\ \}\ {}_\circ n_\alpha = 0$$

and

$$\frac{t_c}{2}\{(\tilde{S}'_3)_h + (\tilde{S}''_3)_{-h}\} + \tilde{t}_3 - [\ \frac{t_c}{2}\{N'_{\alpha\beta}(U_{3,\beta} - \frac{t_c}{2}\Psi_{3,\beta}) + Q'_\alpha\} + M_{\alpha\beta}U_{3,\beta}$$

$$+ K_{\alpha\beta}\Psi_{3,\beta} + T_\alpha + \frac{t_c}{2}\{N''_{\alpha\beta}(U_{3,\beta} + \frac{t_c}{2}\Psi_{3,\beta}) + Q''_\alpha\ \}]\ {}_\circ n_\alpha = 0 \quad (39)$$

(2) On ${}_\circ C_V$:

Boundary conditions on ${}_\circ C_V$ are exactly the same as those on ${}_\circ C_S$ except that the tildes (∿) are removed from all the quantities in which they appear. In addition, the following conditions hold on ${}_\circ C_V$:

$$U'_\alpha = \tilde{U}'_\alpha,\ U_\alpha = \tilde{U}_\alpha,\ U''_\alpha = \tilde{U}''_\alpha,\ U_3 = \tilde{U}_3,\ \Psi_\alpha = \tilde{\Psi}_\alpha,\ \Psi_3 = \tilde{\Psi}_3 \quad (40)$$

(3) On ${}_\circ A_V$:

$$U'_\alpha = \tilde{U}'_\alpha,\ U_\alpha = \tilde{U}_\alpha,\ U''_\alpha = \tilde{U}''_\alpha,\ U_3 = \tilde{U}_3,\ \Psi_\alpha = \tilde{\Psi}_\alpha,\ \Psi_3 = \tilde{\Psi}_3 \quad (41)$$

Strain Functions in Terms of Displacement Functions

$${}_{o}\gamma'_{\alpha\beta} = \tfrac{1}{2}(U'_{\alpha,\beta} + U'_{\beta,\alpha} + U'_{3,\alpha}\, U'_{3,\beta})$$

$${}_{1}\gamma'_{\alpha\beta} = \tfrac{1}{2}(\Psi'_{\alpha,\beta} + \Psi'_{\beta,\alpha})$$

$${}_{o}\gamma''_{\alpha\beta} = \tfrac{1}{2}(U''_{\alpha,\beta} + U''_{\beta,\alpha} + U''_{3,\alpha}\, U''_{3,\beta})$$

$${}_{1}\gamma''_{\alpha\beta} = \tfrac{1}{2}(\Psi''_{\alpha,\beta} + \Psi''_{\beta,\alpha})$$

$${}_{o}\gamma_{\alpha\beta} = \tfrac{1}{2}(U_{\alpha,\beta} + U_{\beta,\alpha} + U_{3,\alpha} U_{3,\beta})$$

$${}_{1}\gamma_{\alpha\beta} = \tfrac{1}{2}(\Psi_{\alpha,\beta} + \Psi_{\beta,\alpha} + U_{3,\alpha}\, \Psi_{3,\beta} + U_{3,\beta}\Psi_{3,\alpha})$$

$${}_{2}\gamma_{\alpha\beta} = \tfrac{1}{2}\Psi_{3,\alpha}\, \Psi_{3,\beta}$$

$${}_{o}\gamma'_{\alpha 3} = \tfrac{1}{2}(\Psi'_{\alpha} + U'_{3,\alpha})$$

$${}_{o}\gamma''_{\alpha 3} = \tfrac{1}{2}(\Psi''_{\alpha} + U''_{3,\alpha})$$

$${}_{o}\gamma_{\alpha 3} = \tfrac{1}{2}(\Psi_{\alpha} + U_{3,\alpha})$$

$${}_{1}\gamma_{\alpha 3} = \tfrac{1}{2}\Psi_{3,\alpha}$$

$${}_{o}\gamma_{33} = \Psi_{3}$$

and

$${}_n\gamma'_{\alpha\beta} = {}_n\gamma''_{\alpha\beta} = {}_n\gamma_{\alpha 3} = 0 \quad \text{for } n \geq 2$$

$${}_n\gamma_{\alpha 3} = 0 \quad \text{for } n \geq 3$$

$${}_n\gamma'_{\alpha 3} = {}_n\gamma''_{\alpha 3} = {}_n\gamma_{33} = 0 \quad \text{for } n \geq 1$$

$${}_n\gamma'_{33} = {}_n\gamma''_{33} = 0 \quad \text{for all values of } n \tag{42}$$

Relations Between Various Stress-Resultants and the Strain Functions

$$N'_{\alpha\beta} = \frac{\partial \varepsilon'}{\partial_0 \gamma'_{\alpha\beta}}, \quad M'_{\alpha\beta} = \frac{\partial \varepsilon'}{\partial_1 \gamma'_{\alpha\beta}}, \quad Q'_{\alpha} = \frac{\partial \varepsilon'}{\partial_0 \gamma'_{\alpha 3}}$$

$$N''_{\alpha\beta} = \frac{\partial \varepsilon''}{\partial_0 \gamma''_{\alpha\beta}}, \quad M''_{\alpha\beta} = \frac{\partial \varepsilon''}{\partial_1 \gamma''_{\alpha\beta}}, \quad Q''_{\alpha} = \frac{\partial \varepsilon''}{\partial_0 \gamma''_{\alpha 3}}$$

$$N_{\alpha\beta} = \frac{\partial \varepsilon}{\partial_0 \gamma_{\alpha\beta}}, \quad M_{\alpha\beta} = \frac{\partial \varepsilon}{\partial_1 \gamma_{\alpha\beta}}, \quad K_{\alpha\beta} = \frac{\partial \varepsilon}{\partial_2 \gamma_{\alpha\beta}}$$

$$Q_{\alpha} = \frac{\partial \varepsilon}{\partial_0 \gamma_{\alpha 3}}, \quad T_{\alpha} = \frac{\partial \varepsilon}{\partial_1 \gamma_{\alpha 3}}, \quad N_{33} = \frac{\partial \varepsilon}{\partial_0 \gamma_{33}} \tag{43}$$

Equations (38) through (43) constitute a complete set of equations necessary and sufficient to solve, in principle, any mixed boundary value problem of sandwich plates.

The equilibrium Equation (38) in a slightly more general form (because of using the exact Equation (4) instead of the approximate Equation (3) which has been used in the present analysis) were obtained by Ebcioglu [20] using a different approach. The reason for choosing the approximate relations (3) instead of the exact relations (4) has been explained before. Also, the present analysis could be carried out using the exact Equation (4) without any difficulty.

The advantages of using the present analysis over the one by Ebcioglu [20] are: (a) formulation of a more general mixed-boundary value problem, and (b) a unified treatment of a general sandwich plate problem. This means that the strain functions in terms of the displacement functions given by Equation (42) and the stress resultants in terms of the strain functions given by Equation (43) are obtained as results of the present analysis along with the equilibrium equations and boundary conditions, unlike the analysis given in [20]. Thus the present analysis can be considered to be a more general one than the previous work by Ebcioglu.

## Constitutive Equations and Stress Resultants in Terms of Displacement Functions for Isotropic Material

The analysis presented so far does not consider any specific constitutive equation (i.e., stress-strain relationship) and is therefore valid for any kind of elastic material (like non-homogeneous and orthotropic). This section considers a sandwich plate whose faces and cores are made of isotropic and homogeneous materials.

Both the materials are made of the same material having shear modulus and Poisson's ratio equal to $G_f$ and $\nu_f$ respectively, and the core made of a different material having shear modulus and Poisson's ratio equal to $G_c$ and $\nu_c$ respectively.

For a linearly isotropic material, the constitutive equations are:

$$S_{ij} = \lambda \gamma_{kk} \delta_{ij} + 2\mu \gamma_{ij} \tag{44}$$

where $\lambda$ and $\mu$ are Lamé-constants, and

$$\lambda = \frac{2\nu G}{1-2\nu}$$

$$\mu = G \tag{45}$$

Substituting (45) into (44), the following result is obtained:

$$S_{ij} = G(\frac{2\nu}{1-2\nu} \gamma_{kk} \delta_{ij} + 2\gamma_{ij}) \tag{46}$$

Also,

$$\gamma_{kk} = \gamma_{\mu\mu} + \gamma_{33}$$

Equation (46) can also be written as:

$$S_{\alpha\beta} = G[\frac{2\nu}{1-2\nu}(\gamma_{\mu\mu} + \gamma_{33})\delta_{\alpha\beta} + 2\gamma_{\alpha\beta}]$$

$$S_{\alpha 3} = 2G\gamma_{\alpha 3}$$

$$S_{33} = \frac{2G}{1-2\nu}[\nu\gamma_{\mu\mu} + (1-\nu)\gamma_{33}] \tag{47}$$

In the faces, the transverse normal stresses are negligibly small, i.e., $S_{33} \simeq 0$ which gives, by (47),

$$\gamma_{33} = -\frac{\nu\gamma_{\mu\mu}}{1-\nu}$$

and when this is substituted in the first set of equations (47), the following result is obtained:

$$S_{\alpha\beta} = G[\frac{2\nu}{1-\nu}\gamma_{\mu\mu}\delta_{\alpha\beta} + 2\gamma_{\alpha\beta}]$$

$$S_{\alpha 3} = 2G\gamma_{\alpha 3}$$

$$S_{33} = 0 \tag{48}$$

For each layer, Equations (17) are first substituted into the strain-displacement relations (3), the Equations (3) are then substituted into (48) to get the expressions for the stresses, and finally these stresses are substituted into (24) to get the stress-resultants in terms of the displacement functions. The results are:

$$N'_{\alpha\beta} = t_f G_f [\frac{\nu f}{1-\nu_f} (2U_{\mu,\mu} + t_c \Psi_{\mu,\mu} + t_f \Psi'_{\mu,\mu} + U_{3,\mu} U_{3,\mu} + t_c U_{3,\mu} \Psi_{3,\mu}$$

$$+ \frac{t_c{}^2}{4} \Psi_{3,\mu} \Psi_{3,\mu}) \delta_{\alpha\beta} + \{U_{\alpha,\beta} + U_{\beta,\alpha} + \frac{t_c}{2} (\Psi_{\alpha,\beta} + \Psi_{\beta,\alpha}) + \frac{t_f}{2} (\Psi'_{\alpha,\beta} + \Psi'_{\beta,\alpha})$$

$$+ U_{3,\alpha} U_{3,\beta} + \frac{t_c}{2} (U_{3,\beta} \Psi_{3,\alpha} + U_{3,\alpha} \Psi_{3,\beta}) + \frac{t_c{}^2}{4} \Psi_{3,\alpha} \Psi_{3,\beta}\}]$$

$$M'_{\alpha\beta} = \frac{t_f{}^3 G_f}{12} [\frac{2\nu_f}{1-\nu_f} \Psi'_{\mu,\mu} \delta_{\alpha\beta} + (\Psi'_{\alpha,\beta} + \Psi'_{\beta,\alpha})]$$

$$Q'_\alpha = t_f G_f (\Psi'_\alpha + U_{3,\alpha} + \frac{t_c}{2} \Psi_{3,\alpha})$$

$$N''_{\alpha\beta} = t_f G_f [\frac{\nu f}{1-\nu_f} (2U_{\mu,\mu} - t_c \Psi_{\mu,\mu} - t_f \Psi''_{\mu,\mu} + U_{3,\mu} U_{3,\mu} - t_c U_{3,\mu} \Psi_{3,\mu} +$$

$$\frac{t_c{}^2}{4} \Psi_{3,\mu} \Psi_{3,\mu}) \delta_{\alpha\beta} + \{U_{\alpha,\beta} + U_{\beta,\alpha} - \frac{t_c}{2} (\Psi_{\alpha,\beta} + \Psi_{\beta,\alpha}) - \frac{t_f}{2} (\Psi''_{\alpha,\beta} + \Psi''_{\beta,\alpha}) + U_{3,\alpha} U_{3,\beta}$$

$$- \frac{t_c}{2} (U_{3,\beta} \Psi_{3,\alpha} + U_{3,\alpha} \Psi_{3,\beta}) + \frac{t_c{}^2}{4} \Psi_{3,\alpha} \Psi_{3,\beta}\}]$$

$$M''_{\alpha\beta} = \frac{t_f{}^3 G_f}{12} [\frac{2\nu f}{1-\nu_f} \Psi''_{\mu,\mu} \delta_{\alpha\beta} + (\Psi''_{\alpha,\beta} + \Psi''_{\beta,\alpha})]$$

$$Q''_\alpha = t_f G_f (\Psi''_\alpha + U_{3,\alpha} - \frac{t_c}{2} \Psi_{3,\alpha})$$

$$N_{\alpha\beta} = G_c [ \{\frac{2\nu_c}{1-2\nu_c} (U_{\mu,\mu} + \tfrac{1}{2} U_{3,\mu} U_{3,\mu} + \Psi_3) \delta_{\alpha\beta} + U_{\alpha,\beta} + U_{\beta,\alpha} + U_{3,\alpha} U_{3,\beta}\} t_c$$

(contd.)

$$+ \left( \frac{\nu_c}{1-2\nu_c} \Psi_{3,\mu}\Psi_{3,\mu}\, \delta_{\alpha\beta} + \Psi_{3,\alpha}\Psi_{3,\beta} \right) \frac{t_c^{\,3}}{12} ]$$

$$M_{\alpha\beta} = \frac{G_c t_c^{\,3}}{1-2\nu_c} [\frac{2\nu_c}{1-2\nu_c} (\Psi_{\mu,\mu} + U_{3,\mu}\Psi_{3,\mu})\delta_{\alpha\beta} + (\Psi_{\alpha,\beta} + \Psi_{\beta,\alpha} + U_{3,\beta}\Psi_{3,\alpha} + U_{3,\alpha}\Psi_{3,\beta})] \frac{t_c^{\,3}}{12}$$

$$K_{\alpha\beta} = G_c[\{\frac{2\nu_c}{1-2\nu_c} (U_{\mu,\mu} + \tfrac{1}{2}U_{3,\mu}U_{3,\mu} + \Psi_3)\delta_{\alpha\beta} + U_{\alpha,\beta} + U_{\beta,\alpha} + U_{3,\alpha}U_{3,\beta}\} \frac{t_c^{\,3}}{12}$$

$$+ \left( \frac{\nu_c}{1-2\nu_c} \Psi_{3,\mu}\Psi_{3,\mu}\delta_{\alpha\beta} + \Psi_{3,\alpha}\Psi_{3,\beta} \right) \frac{t_c^{\,5}}{80} ]$$

$$Q_\alpha = t_c G_c (\Psi_\alpha + U_{3,\alpha})$$

$$T_\alpha = \frac{t_c^{\,3}}{12} G_c \Psi_{3,\alpha}$$

$$N_{33} = \frac{2G_c}{1-2\nu_c} [\{ \frac{\nu_c}{2} (2U_{\mu,\mu} + U_{3,\mu}U_{3,\mu}) + (1-\nu_c)\Psi_3\} t_c + \frac{\nu_c}{24} \Psi_{3,\mu}\Psi_{3,\mu} t_c^{\,3}] \tag{49}$$

Substitution of Equation (49) into Equations (38) through (43) results in obtaining a complete formulation of a general and nonlinear problem of sandwich plates in terms of displacement functions.

# CHAPTER 3

# STABILITY ANALYSIS OF ELASTIC SANDWICH PLATES

## Combined Axial and Transverse Loads and the Problem of Buckling

A special case of the analysis presented in Chapter 2 will be treated here. ${}_{\circ}A$ is now considered to be consisting wholly of ${}_{\circ}A_s$ alone, and the given surface forces over ${}_{\circ}A_s$ are all acting perpendicularly to the plate. Also, the body forces, moments and the acceleration forces and moments are negligible compared to other forces. Then,

$$(\tilde{S}'_{\alpha *})_{-h} = (\tilde{S}''_{\alpha *})_{h} = 0$$

$$F'_{\alpha} = f'_{\alpha} = F''_{\alpha} = f''_{\alpha} = F_{\alpha} = f_{\alpha} = F'_3 = f'_3 = F''_3 = f''_3 = F_3 = f_3 = 0$$

and

$$M'_{\alpha} = m'_{\alpha} = M''_{\alpha} = m''_{\alpha} = M_{\alpha} = m_{\alpha} = M'_3 = m'_3 = M''_3 = m''_3 = M_3 = m_3 = 0 \quad (50)$$

The axial stress resultants $N'_{\alpha\beta}$, $N''_{\alpha\beta}$ and $N_{\alpha\beta}$ can be expressed as:

$$N'_{\alpha\beta} = N^{\circ}_{\alpha\beta} + n'_{\alpha\beta}$$

$$N''_{\alpha\beta} = N^{\circ}_{\alpha\beta} + n''_{\alpha\beta}$$

$$N_{\alpha\beta} = \overline{N}_{\alpha\beta} + n_{\alpha\beta} \qquad (51)$$

where $N^{\circ}_{\alpha\beta}$ and $\overline{N}_{\alpha\beta}$ are initial axial stresses in the face layers and in the core respectively before transverse deflection occurs, and $n'_{\alpha\beta}$, $n''_{\alpha\beta}$ and $n_{\alpha\beta}$ are additional small stresses in the upper face, lower face and the core respectively, giving rise to buckling.

It is assumed that:

$$n'_{\alpha\beta},\ n''_{\alpha\beta} << N^{\circ}_{\alpha\beta}$$

and $$n_{\alpha\beta} << \overline{N}_{\alpha\beta} \qquad (52)$$

i.e., $n'_{\alpha\beta}$ and $n''_{\alpha\beta}$ are much smaller than $N^{\circ}_{\alpha\beta}$ and $n_{\alpha\beta}$ are much smaller than $\overline{N}_{\alpha\beta}$. The values of $n'_{\alpha\beta}$ and $n''_{\alpha\beta}$ are equal to the values of $N'_{\alpha\beta}$ and $N''_{\alpha\beta}$ respectively after dropping all the nonlinear terms.

From (50) and (38), the following equilibrium equations are obtained:

$$N'_{\alpha\beta,\beta} + \frac{2}{t_f}(M'_{\alpha\beta,\beta} - Q'_{\alpha}) = 0 \qquad (53)$$

$$N''_{\alpha\beta,\beta} - \frac{2}{t_f}(M''_{\alpha\beta,\beta} - Q''_{\alpha}) = 0 \qquad (54)$$

$$\frac{2}{t_f}(Q'_{\alpha} - M'_{\alpha\beta,\beta} - Q''_{\alpha} + M''_{\alpha\beta,\beta}) + N_{\alpha\beta,\beta} = 0 \qquad (55)$$

$$\frac{t_c}{t_f}\left(Q'_\alpha - M'_{\alpha\beta,\beta} + Q''_\alpha - M''_{\alpha\beta,\beta}\right) + M_{\alpha\beta,\beta} - Q_\alpha = 0 \tag{56}$$

$$\{(N'_{\alpha\beta}+N''_{\alpha\beta}+N_{\alpha\beta})U_{3,\beta}\}_{,\alpha} + \{(N'_{\alpha\beta}-N''_{\alpha\beta})\,\frac{t_c}{2}\,\Psi_{3,\beta} + M_{\alpha\beta}\Psi_{3,\beta}\}_{,\alpha}$$

$$+(Q'_\alpha + Q''_\alpha + Q_\alpha)_{,\alpha} + (\tilde{S}'_{3*})_h + (\tilde{S}''_{3*})_{-h} = 0 \tag{57}$$

$$\frac{t_c}{2}\,\{(N'_{\alpha\beta}-N''_{\alpha\beta})\;U_{3,\beta}\}_{,\alpha} + \frac{t_c{}^2}{4}\;(N'_{\alpha\beta}+N''_{\alpha\beta})\Psi_{3,\beta}\}_{,\alpha} + \frac{t_c}{2}\,(Q'_\alpha - Q''_\alpha)_{,\alpha}$$

$$+ (M_{\alpha\beta}U_{3,\beta}+K_{\alpha\beta}\Psi_{3,\beta}+T_\alpha)_{,\alpha} - N_{33} + \frac{t_c}{2}\,\{(\tilde{S}'_{3*})_h - (\tilde{S}''_{3*})_{-h}\} = 0 \tag{58}$$

From (53) and (54), the following relations are obtained:

$$\frac{2}{t_f}\,(Q'_\alpha - M'_{\alpha\beta,\beta}) = N'_{\alpha\beta,\beta}$$

$$\frac{2}{t_f}\,(M''_{\alpha\beta,\beta} - Q''_\alpha) = N''_{\alpha\beta,\beta} \tag{59}$$

Then (59) is substituted into (55) and the result is:

$$N'_{\alpha\beta,\beta} + N''_{\alpha\beta,\beta} + N_{\alpha\beta,\beta} = 0 \tag{60}$$

Substitution of (59) into (56) gives:

$$\frac{2}{t_c}\,(M_{\alpha\beta,\beta} - Q_\alpha) + (N'_{\alpha\beta} - N''_{\alpha\beta})_{,\beta} = 0 \tag{61}$$

Using Equation (60) in (57) the following equation is obtained:

$$(N'_{\alpha\beta} + N''_{\alpha\beta} + N_{\alpha\beta})\, U_{3,\beta\alpha} + \{(N'_{\alpha\beta} - N''_{\alpha\beta})\, \frac{t_c}{2}\, \Psi_{3,\beta} + M_{\alpha\beta}\Psi_{3,\beta}\}_{,\alpha}$$

$$+ (Q'_\alpha + Q''_\alpha + Q_\alpha)_{,\alpha} + (\tilde{S}'_{3*})_h + (\tilde{S}''_{3*})_{-h} = 0 \qquad (62)$$

Subtraction of (54) from (53) gives:

$$\frac{2}{t_f}\,[(M'_{\alpha\beta} + M''_{\alpha\beta})_{,\beta} - (Q'_\alpha + Q''_\alpha)] + (N'_{\alpha\beta} - N''_{\alpha\beta})_{,\beta} = 0 \qquad (63)$$

Addition of (53) and (54) gives:

$$\frac{2}{t_f}\,[(M'_{\alpha\beta} - M''_{\alpha\beta})_{,\beta} - (Q'_\alpha - Q''_\alpha)] + (N'_{\alpha\beta} + N''_{\alpha\beta})_{,\beta} = 0 \qquad (64)$$

Now, for small deformations beyond the critical point in the transverse direction the middle surface of the core is assumed to remain undeformed, which means:

$$n_{\alpha\beta} = 0$$

Then, from (51) it follows that:

$$N_{\alpha\beta} = \overline{N}_{\alpha\beta}$$

Substituting (51) into (61), (63), (64), (62) and (58), and making use of (60), the following equations are obtained:

$$\frac{2}{t_c}(M_{\alpha\beta,\beta} - Q_\alpha) + (n'_{\alpha\beta} - n''_{\alpha\beta})_{,\beta} = 0 \tag{65}$$

$$\frac{2}{t_f}[(M'_{\alpha\beta} + M''_{\alpha\beta})_{,\beta} - (Q'_\alpha + Q''_\alpha)] + (n'_{\alpha\beta} - n''_{\alpha\beta})_{,\beta} = 0 \tag{66}$$

$$\frac{2}{t_f}[(M'_{\alpha\beta} - M''_{\alpha\beta})_{,\beta} - (Q'_\alpha - Q''_\alpha)] + (n'_{\alpha\beta} + n''_{\alpha\beta})_{,\beta} = 0 \tag{67}$$

$$(Q'_\alpha+Q''_\alpha+Q_\alpha)_{,\alpha} + (2N^\circ_{\alpha\beta}+N_{\alpha\beta})\,U_{3,\alpha\beta} + \{(n'_{\alpha\beta}-n''_{\alpha\beta})\frac{t_c}{2}\Psi_{3,\beta} + M_{\alpha\beta}\Psi_{3,\beta}\}_{,\alpha}$$
$$+(\tilde{S}'_{3*})_h + (\tilde{S}''_{3*}) = 0 \tag{68}$$

and

$$\frac{t_c}{2}\{(n'_{\alpha\beta}-n''_{\alpha\beta})\,U_{3,\beta}\}_{,\alpha} + \frac{t_c^2}{4}(2N^\circ_{\alpha\beta}+n'_{\alpha\beta}+n''_{\alpha\beta})_{,\alpha}\Psi_{3,\beta} + \frac{t_c^2}{2}N^\circ_{\alpha\beta}\Psi_{3,\alpha\beta}$$
$$+\frac{t_c^2}{4}(n'_{\alpha\beta}+n''_{\alpha\beta})\,\Psi_{3,\alpha\beta} + \frac{t_c}{2}(Q'_\alpha-Q''_\alpha)_{,\alpha} + (M_{\alpha\beta}U_{3,\beta}+K_{\alpha\beta}\Psi_{3,\beta}+T_\alpha)_{,\alpha}$$
$$-N_{33} + \frac{t_c}{2}\{(\tilde{S}'_{3*})_h - (\tilde{S}''_{3*})_{-h}\} = 0 \tag{69}$$

After dropping all the nonlinear terms from Equations (68) and (69) and making use of (60), the Equations (68) and (69) become:

$$(Q'_\alpha+Q''_\alpha+Q_\alpha)_{,\alpha} + (2N^\circ_{\alpha\beta}+N_{\alpha\beta})\,U_{3,\alpha\beta} + (\tilde{S}'_{3*})_h + (\tilde{S}''_{3*})_{-h} = 0 \tag{70}$$

and

$$\frac{2}{t_c}(T_{\alpha,\alpha}-N_{33}) + (Q'_\alpha-Q''_\alpha)_{,\alpha} + t_c N^\circ_{\alpha\beta}\,\Psi_{3,\alpha\beta} + (\tilde{S}'_{3*})_h - (\tilde{S}''_{3*})_{-h} = 0 \quad (71)$$

In a buckling problem in which deflections in the transverse direction are small, the displacements of the middle plane of the core in the direction of $x_\alpha$ can be neglected because the said displacements are very small compared to their values just prior to the occurrence of buckling. Therefore,

$$U_\alpha = 0 \quad (72)$$

This reduces the number of unknown quantities from ten to eight for the buckling problem. Consequently, there remains no need to consider all the ten equations given by (55), (65), (66), (67), (70), and (71). The two equations given by (55) can be shown to be redundant and are therefore dropped. The remaining equations are:

$$\frac{2}{t_c}(M_{\alpha\beta,\beta} - Q_\alpha) + (n'_{\alpha\beta} - n''_{\alpha\beta})_{,\beta} = 0 \quad (73)$$

$$\frac{2}{t_f}[(M'_{\alpha\beta} + M''_{\alpha\beta})_{,\beta} - (Q'_\alpha + Q''_\alpha)] + (n'_{\alpha\beta} - n''_{\alpha\beta})_{,\beta} = 0 \quad (74)$$

$$\frac{2}{t_f}[(M'_{\alpha\beta} - M''_{\alpha\beta})_{,\beta} - (Q'_\alpha - Q''_\alpha)] + (n'_{\alpha\beta} + n''_{\alpha\beta})_{,\beta} = 0 \quad (75)$$

$$(Q'_\alpha+Q''_\alpha+Q_\alpha)_{,\alpha} + (2N^\circ_{\alpha\beta}+N_{\alpha\beta})\,w_{,\alpha\beta} + (\underline{p}' + \underline{p}'') = 0 \quad (76)$$

and

$$\frac{2}{t_c}(T_{\alpha,\alpha}-N_{33}) + (Q'_{\alpha}-Q''_{\alpha})_{,\alpha} + 2N^{\circ}_{\alpha\beta}W_{,\alpha\beta} + (\underline{p}' - \underline{p}'') = 0 \tag{77}$$

where

$$w = u_3$$

$$W = \frac{t_c}{2}\Psi_3$$

$$\underline{p}' = (S'_{3*})_h$$

and $$\underline{p}'' = (S''_{*})_{-h} \tag{78}$$

Here, w is the transverse deflection of the middle surface of the core and is therefore the antisymmetric or overall deflection of the sandwich plate. W is the transverse deflection of either face from the middle surface of the core, and is therefore the symmetric deflection of the sandwich plate. The quantities $\underline{p}'$ and $\underline{p}''$ are the loads acting perpendicularly to the upper and lower surfaces respectively and are positive when both of them act in the $x_3$ direction.

The simplified Equations (73) through (77) were obtained by Kim [42] in his linear analysis for the combined axial and transverse loads problem. The present analysis is therefore a general treatment of which Kim's equations can be obtained as a special case.

### Kim's Solution of the Problem of Combined Axial and Transverse Loads

The material presented from here on up to the end of this chapter is a summary of results obtained by Kim [42] in deriving the expressions for the antisymmetric and symmetric deflections (w and W) of an elastic sandwich plate from the Equations (73) through (78). These expressions for w and W for an elastic sandwich plate will then be used to derive the time dependent expressions of the same two quantities for a viscoelastic sandwich plate making use of the Correspondence Principle of the Viscoelastic Theory.

The following quantities are defined similar to the notations used in [42] and [32]:

$$u'_\alpha = \frac{t_f}{2} \Psi'_\alpha$$

$$u''_\alpha = \frac{t_f}{2} \Psi''_\alpha$$

$$u_\alpha = \frac{t_c}{2} \Psi_\alpha$$

$$\tilde{f}_{\alpha\beta} = \frac{1}{2G_f t_f} (2N^\circ_{\alpha\beta} + N_{\alpha\beta})$$

$$\hat{f}_{\alpha\beta} = \frac{1}{G_f t_f} N^{\circ}_{\alpha\beta}$$

$$\tilde{q} = \frac{1}{2G_f t_f} (\underline{p}' + \underline{p}'')$$

$$\hat{q} = \frac{1}{2G_f t_f} (p' - p'')$$

$$g = \frac{G_c t_c}{G_f t_f}$$

$$\tilde{u}_\alpha = \tfrac{1}{2}(U'_\alpha + U''_\alpha)$$

and $$\hat{u}_\alpha = \tfrac{1}{2}(U'_\alpha - U''_\alpha) \qquad (79)$$

First $u'_\alpha$ and $u''_\alpha$ are solved in terms of $\tilde{u}_\alpha$, $\hat{u}_\alpha$ and the results are substituted into the expressions for $\Psi'_\alpha$ and $\Psi''_\alpha$ in the above equations. Then the various stress resultants given in (49) are expressed in terms of the $\tilde{u}_\alpha$, $\hat{u}_\alpha$, $u_\alpha$, $\hat{f}_{\alpha\beta}$, $\hat{f}$, $\tilde{q}$, $\hat{q}$, g, w and W defined in (78) and (79). The non-linear terms and the quantities $u_\alpha$ are then dropped from these expressions. Keeping in mind that the quantities $n'_{\alpha\beta}$ and $n''_{\alpha\beta}$ are given by the linearized expressions of $N'_{\alpha\beta}$ and $N''_{\alpha\beta}$ respectively, Equations (49) are then substituted into Equations (73) through (77). Thus the equilibrium equations in terms of the displacement functions and initial stress resultantsare obtained as follows:

$$\frac{g}{6}\left(\frac{1}{1-2\nu_c} u_{\beta,\beta\alpha} + u_{\alpha,\beta\beta}\right) - \frac{g}{t_c}\left(\frac{2}{t_c} u_\alpha + w,_\alpha\right) + \frac{1+\nu_f}{1-\nu_f}\left(u_{\beta,\beta\alpha} + \tilde{u}_{\beta,\beta\alpha}\right)$$

$$+ u_{\alpha,\beta\beta} + \tilde{u}_{\alpha,\beta\beta} = 0 \tag{80}$$

$$\frac{1+\nu_f}{1-\nu_f}\left(u_{\beta,\beta\alpha} + \frac{4}{3}\tilde{u}_{\beta,\beta\alpha}\right) + u_{\alpha,\beta\beta} + \frac{4}{3}\tilde{u}_{\alpha,\beta\beta} - \frac{2}{t_f}\left(\frac{2}{t_f}\tilde{u}_\alpha + w,_\alpha\right) = 0 \tag{81}$$

$$\frac{2}{t_f}\tilde{u}_{\alpha,\alpha} + w,_{\alpha\alpha} + \frac{g}{2}\left(\frac{2}{t_c} u_{\alpha,\alpha} + w,_{\alpha\alpha}\right) + \tilde{f}_{\alpha\beta} w,_{\alpha\beta} + \tilde{q} = 0 \tag{82}$$

$$\frac{1+\nu_f}{1-\nu_f} u_{\beta,\beta\alpha} + u_{\alpha,\beta\beta} - \frac{3}{2t_f}\left(\frac{2}{t_f}\tilde{u}_\alpha + W,_\alpha\right) = 0 \tag{83}$$

$$\left(\frac{2}{t_f} u_{\alpha,\alpha} \; W,_{\alpha\alpha}\right) + \frac{g}{6} W,_{\alpha\alpha} - \frac{4g(1-\nu_c)}{t_c^{\,2}(1-2\nu_c)} W + f_{\alpha\beta} W,_{\alpha\beta} + \hat{q} = 0 \tag{84}$$

There are eight equations and eight unknowns in (80) through (84); hence, they can be completely solved with the appropriate boundary conditions. For a rectangular plate simply supported on all sides, the solution is given in [42].

However, in a sandwich plate problem, it is usually of interest to calculate w and W only. For that reason, eliminating $u_\alpha$ and $\tilde{u}_\alpha$ from Equations (80), (81) and (82), a single equation in w is obtained. Similarly, eliminating $\tilde{u}_\alpha$ from (83) and (84), a single equation in W is obtained.

They are as follows:

$$- R_1\nabla^6 w + R_2\nabla^4 w = (1-R_3\nabla^2 + R_4\nabla^4)\ [(2N^\circ_{\alpha\beta} + N_{\alpha\beta})\ w,_{\alpha\beta} + \underline{p}'+\underline{p}''] \qquad (85)$$

and

$$4D_f\ (1 + \frac{g}{6})\ \nabla^4 W - \frac{G_c t_c}{6}\ [1 + \frac{16t_f{}^2(1-\nu_c)}{t_c{}^2(1-2\nu_c)(1-\nu_c)}\ ]\ \nabla^2 W\ + \frac{4G_c(1-\nu_c)}{t_c(1-2\nu_c)}\ W$$

$$= \tfrac{1}{2}[1- \frac{2t_f{}^2}{3(1-\nu_f)}\ \nabla^2]\ (2N^\circ_{\alpha\beta}W,_{\alpha\beta} + \underline{p}' - \underline{p}'') \qquad (86)$$

where

$$R_1 = \frac{2D_f D_1}{G_c t_c}\ (1 + \frac{g}{2})\ [1 + \frac{2g(1-\nu_c)(1-\nu_f)}{3(1-2\nu_c)}$$

$$R_2 = D_1[1 + \frac{2t_f}{t_c} + \frac{4t_f{}^2}{3t_c{}^2} + \frac{g(1-\nu_c)(1-\nu_f)}{6(1-2\nu_c)}\ ]$$

$$R_3 = \frac{D_1}{G_c t_c}\ [1 + g\ \{\frac{2t_f{}^2}{3t_c{}^2} + \frac{(1-\nu_c)(1-\nu_f)}{6(1-2\nu_c)}\}\ ]$$

$$R_4 = \frac{D_f D_1}{G_f G_c t_f t_c}\ [1 + \frac{2g(1-\nu_c)(1-\nu_f)}{3(1-2\nu_c)}\ ]$$

$$D_1 = \frac{G_f t_f t_c{}^2}{1-\nu_f}$$

$$D_f = \frac{E_f t_f^{\,3}}{12(1-\nu_f^{\,2})}$$

and

$E_f$ = Young's Modulus of the face layers. (87)

The order of the differential Equation (85) is six, and that of (86) is four, therefore these two equations require three conditions and two conditions respectively at each side. Since the number of available boundary conditions for w and W is one each at each of the four sides of the panel, in general there are not enough boundary conditions to use Equations (85) and (86) independently.

## Rectangular Sandwich Plate Simply Supported on All Sides

A rectangular sandwich plate having sides of length a and b along x and y directions respectively is considered. Let $x = x_1$, $y = x_2$ and $z = x_3$. The origin of the coordinate axes is located at one of the corners of the middle surface of the core. The plate is subjected to arbitrary transverse loads $\underline{p}'$ and $\underline{p}''$, and a uniformly distributed axial compressive load N along the face edges x = 0 and x = a. Then,

$$N_x^\circ = -N$$

$$N_{xy}^\circ = N_y^\circ = N_x = N_{xy} = N_y = 0 \qquad (88)$$

where

$$N_x^\circ = N_{xx}, \quad N_y^\circ = N_{yy}, \text{ etc.}$$

Let

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \text{ Sin } K_1 x \text{ Sin } K_2 y$$

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \text{ Sin } K_1 x \text{ Sin } K_2 y$$

$$\tilde{q}(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{q}_{mn} \text{ Sin } K_1 x \text{ Sin } K_2 y$$

and

$$\hat{q}(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \hat{q}_{mn} \text{ Sin } K_1 x \text{ Sin } K_2 y \tag{89}$$

where

$$K_1 = \frac{m\pi}{a}; \quad K_2 = \frac{m\pi}{b}$$

$$\tilde{q}_{mn} = \frac{4}{ab} \int_0^a \int_0^b \tilde{q}(x, y) \text{ Sin } K_1 x \text{ Sin } K_2 y \, dx \, dy$$

and

$$\hat{q}_{mn} = \frac{4}{ab} \int_0^a \int_0^b \hat{q}(x, y) \text{ Sin } K_1 x \text{ Sin } K_2 y \, dx \, dy \tag{90}$$

If Equations (89) are substituted into Equations (85) and (86), they are found to be satisfied. The boundary conditions for various cases have been discussed elaborately in [42] and for the present case it has been shown [42] that if the expressions given in (89) for w and W are used, all the boundary conditions are taken care of.

The values of the coefficients $w_{mn}$ and $W_{mn}$ are obtained by substituting (89) in Equations (85 and (86), and are given as:

$$w_{mn} = \frac{\tilde{q}_{mn}}{\tilde{\Delta}_{mn}} \left[1 + B_3 \left(K_1^2 + K_2^2\right) + B_4 \left(K_1^2 + K_2^2\right)^2 \right] \tag{91}$$

and

$$W_{mn} = \frac{\hat{q}_{mn}}{\hat{\Delta}_{mn}} \left[1 + B_8 \left(K_1^2 + K_2^2\right)\right] \tag{92}$$

where

$$\tilde{\Delta}_{mn} = B_1(K_1^2+K_2^2)^3 + B_2(K_1^2+K_2^2)^2 - [1+B_3(K_1^2+K_2^2) + B_4(K_1^2+K_2^2)^2]K_1^2 f \tag{93}$$

$$\hat{\Delta}_{mn} = B_5(K_1^2+K_2^2)^2 + B_6(K_1^2+K_2^2) + B_7 - [1+B_8(K_1^2+K_2^2)]K_1^2 f \tag{94}$$

and

$$B_1 = \frac{\alpha_f t_f^{\,2} t_c^{\,2}}{6g} \left(1 + \frac{g}{2}\right) \left( {}_f + g \frac{\alpha_c}{3}\right)$$

$$B_2 = \frac{t_c^{\,2}}{2} \left[\alpha_f (1 + 2r + \frac{4}{3} r^2) + \frac{g\alpha_c}{12} \right]$$

$$B_3 = \frac{t_c^{\,2}}{g} \left[\alpha_f \left(1 + \frac{2}{3} g r^2\right) + \frac{g\alpha_c}{12} \right]$$

$$B_4 = \frac{\alpha_f t_c^{\,2} t_f^{\,2}}{6g} \left(\alpha_f + \frac{g\alpha_c}{3} \right)$$

$$B_5 = \frac{\alpha_f t_f^{\,2}}{3} \left(2 + \frac{g}{3} \right)$$

$$B_6 = \frac{2}{6} \left(1 + 8\alpha_f \alpha_c r^2\right)$$

$$B_7 = \frac{2g\alpha_c}{t_c^{\,2}}$$

$$B_8 = \frac{2}{3} \alpha_f t_f^{\,2}$$

$$f = \frac{N}{G_f t_f}$$

$$r = \frac{t_f}{t_c}$$

$$\alpha_f = \frac{1}{1 - \nu_f}$$

$$\alpha_c = \frac{2(1-\nu_c)}{1-2\nu_c} \tag{95}$$

The above results will be used in the next chapter where the problem of a viscoelastic sandwich plate will be treated. The analyses presented at the end of the previous chapter and in this chapter are for elastic sandwich plates made of three layers. However, the procedure would be quite similar for plates made of more layers. Also, the results obtained here can be readily reduced to the case of a plate made of a single layer, thin or moderately thick. If all the primed quantities except $(\tilde{S}'_{3*})_h$ and $(S''_{3*})$ are dropped and the quantities $(\tilde{S}'_{3*})_h$ and $(\tilde{S}''_{3*})$ are replaced by a load acting in the transverse direction, the Equations (73) through (77) are reduced to those for a single layer plate given in [54], and, in addition, if the terms $(S'_{3*})_h$ and $(S''_{3*})$ are dropped, the equation for buckling of a thin plate, as given in [55], is obtained.

# CHAPTER 4

# BENDING AND STABILITY OF VISCOELASTIC SANDWICH PLATES

## Method of Approach

A direct method of formulating the problem of any viscoelastic body (plates, discs, shells, etc.) is to start from the equations of motions of a continuous medium, the compatibility conditions and the constitutive equations applicable to the particular viscoelastic material considered. Many authors, [46], [43], [47, [56] and others, have used this approach for plate and plate-type problems.

There remains, however, another method to obtain the equations or solutions, in a rather indirect way, of a large class of problems in linear viscoelasticity from the known equations or solutions of the similar problems in linear elasticity. This is the correspondence principle which will be explained in the next section. This method of finding the solution in linear viscoelasticity is especially useful and convenient when there already exists a known solution to the "corresponding" elastic problem having the same geometric and boundary conditions (with certain restrictions).

Since in a classical field like linear elasticity, thousands of problems have been solved and their solutions are readily available, they can be used to get the solutions of the "corresponding" viscoelastic problems. This later method has been used by [57], [58], [59] and others to solve plate problems and various other types of problems in viscoelasticity. In the present analysis also, this latter method will be used to get the solution of a viscoelastic sandwich plate problem from the known solution of the "corresponding" elastic sandwich plate problem which has been given in the last chapter.

## Correspondence Principle

This principle has been stated, derived and explained by Lee [60]. However, the version given by Pipkin in [61] of the same principle slightly differs from that given in [60]. According to Pipkin, the one given in his book is the exact version of the correspondence principle. Though it can be easily verified that both the versions give the same result, the one given in [61] will be stated below and used thereafter in the present analysis.

If $\overline{F} = f(s) = \int_0^\infty e^{-st} F(t)dt$ = Laplace transform of F

(t = time)

$$\text{and } \overline{\overline{F}} = s\overline{F}, \tag{96}$$

it has been shown in [61] that the constitutive equations and the equations of equilibrium of a linearly viscoelastic material take the following form in the transformed plane:

$$\bar{\bar{\sigma}}_{ij} = \bar{\bar{c}}_{ijkl}\,\bar{\bar{U}}_{k,l}$$

and

$$\bar{\bar{\sigma}}_{ij,j} = -\rho\bar{\bar{f}}_i \qquad (97)$$

where

$\sigma_{ij}$ = stresses

$c_{ijkl}$ = time dependent elastic constants

$U_i$ = displacements

$\rho$ = density

and

$f_i$ = body force components

Now, the equations of an elastic medium are:

$$\sigma_{ij} = C_{ijkl}\,U_{k,l}$$

and

$$\sigma_{ij,j} = -\rho f_i \qquad (98)$$

Comparing (97) and (98) it is found that the s-multiplied Laplace transforms of the time dependent viscoelastic quantities satisfy the time independent elastic equations for a quasi-static case (where inertia effects are neglected due to slow motions).

Suppose there are two bodies of the same size and with the same boundary conditions, but one of them is elastic and the other one viscoelastic, and for a given set of loading and displacement conditions the solution to the problem of the elastic body is known. The solution to the problem of the viscoelastic body, which is now acted by a set of loading having the same spatial distribution as that of the elastic body and changing with time with other quantities like displacements, elastic moduli etc., is obtained as follows:

The s-multiplied transforms of all variables and elastic constants are substituted into the elastic solution and the inversions are performed.

As pointed out in [62], the correspondence principle is not applicable in the following cases: (1) A point on the boundary of a viscoelastic body may be kept free of stress while loads are being applied elsewhere and after some time a device may be applied which makes this point undergo a prescribed displacement. Then neither the surface traction nor the displacement is known at all times $t > 0$, and neither of these functions can be subjected to the Laplace transformation; (2) The boundaries of a viscoelastic body may be changing in the course of time. Melting bodies, bodies solidifying from melts and bodies extruded from orifice belong to this class.

## Deflection of Viscoelastic Sandwich Plate Under Combined Axial and Transverse Loads

In order to use the results (91) and (93) for solving the corresponding viscoelastic problem, the first step would be to consider all the time dependent quantities and separate them from the rest so that finding their Laplace transforms would be easy. Apart from the loads and displacements, other time dependent quantities are the elastic constants $\nu$ and G. However, if the bulk modulus K and the shear modulus G have the same time-dependence, $\nu$ is constant [61]. In the present analysis it would be assumed that $\nu$ is a constant in time partly because this is a fairly valid assumption for many viscoelastic materials and partly in order to make the analysis simpler. The assumption that $\nu$ is a constant has been made by many authors, e.g. [47], [43], [57].

Thus in Equations (93) through (95) the two quantities g and f turn out to be time dependent, and therefore the expressions for $B_1$ through $B_8$ given in (95) are rewritten as follows after some rearrangements of terms:

$$B_1 = \frac{A_1}{g} + A_2 g + A_3$$

$$B_2 = A_4 + A_5 g$$

$$B_3 = \frac{A_6}{g} + A_7$$

$$B_4 = \frac{A_8}{g} + A_9$$

$$B_5 = A_{10} + A_{11}\, g$$

$$B_6 = A_{12}\, g$$

$$B_7 = A_{13}\, g$$

and

$$B_8 = A_{15} \tag{99}$$

where

$$A_1 = A_8 = \frac{\alpha_f{}^2 t_f{}^2 t_c{}^2}{6} \qquad A_9 = \frac{\alpha_f \alpha_c t_c{}^2 t_f{}^2}{18}$$

$$A_2 = \frac{\alpha_f \alpha_c t_f{}^2 t_c{}^2}{36} \qquad A_{10} = \frac{2\alpha_f t_f{}^2}{3}$$

$$A_3 = \frac{\alpha_f t_f^2 t_c^2}{6}\left(\frac{\alpha_c}{3} + \frac{\alpha_f}{2}\right) \qquad A_{11} = \frac{\alpha_f t_f^2}{9}$$

$$A_4 = \tfrac{1}{2} t_c^2 \alpha_f \left(1 + 2r + \frac{4}{3} r^2\right) \qquad A_{12} = \frac{1 + 8\alpha_f \alpha_c r^2}{6}$$

$$A_5 = t_c^2 \alpha_c \qquad A_{13} = \frac{2\alpha_c}{t_c^2}$$

$$A_6 = t_c^2 \alpha_f \qquad A_{14} = \frac{2}{3} \alpha_f t_f^2$$

$$A_7 = \frac{2}{3} t_c^2 r^2 \alpha_f + \frac{t_c^2 \alpha_c}{12} \tag{100}$$

Let the transverse loads have the following form:

$$\underline{p}' = p_1(x,y)F(t)$$

and

$$\underline{p}'' = p_2(x,y)F(t) \tag{101}$$

where

$$F\ (t) = 0 \text{ for } t \leq 0$$

and

$$F\ (t) = \text{an arbitrary function of time} \tag{102}$$

Then from (79),

$$\tilde{q} = H_1 \left(\frac{F}{G_f}\right) \text{ and } \hat{q} = H_2\left(\frac{F}{G_f}\right) \tag{103}$$

where

$$H_1 = \frac{p_1 + p_2}{2t_f} \qquad \text{and} \qquad H_2 = \frac{p_1 - p_2}{2t_f} \tag{104}$$

Substitution of (103) into (90) gives:

$$\tilde{q}_{mn} = H_3 \left(\frac{F}{G_f}\right) \qquad \text{and} \qquad \hat{q}_{mn} = H_4 \left(\frac{F}{G_f}\right) \tag{105}$$

where

$$H_3 = \frac{4}{ab} \int_o^a \int_o^b H_1 \text{ Sin } K_1 x \text{ Sin } K_2 y \, dxdy$$

and

$$H_4 = \frac{4}{ab} \int_o^a \int_o^b H_2 \text{ Sin } K_1 x \text{ Sin } K_2 y \, dxdy \tag{106}$$

Now the Equations (99) are first substituted into (93) and (94), then (93), (94) and (105) are substituted into (91) and (92) to get the expressions for $w_{mn}$ as follows:

$$w_{mn} = \frac{H_3 F \left(H_5 g + H_6\right)}{G_f\{H_7 + H_8 g + H_9 g^2 - (H_{10} g + H_{11}) f\}} \tag{107}$$

and

$$W_{mn} = \frac{H_{12}F}{G_f(H_{13}g + H_{14} - H_{15}g)} \tag{108}$$

where

$$H_5 = 1 + A_7K_3 + A_9K_3^2$$

$$H_6 = A_6K_3 + A_8K_3^2$$

$$H_7 = A_1K_3^3$$

$$H_8 = A_3K_3^3 + A_4K_3^2$$

$$H_9 = A_2K_3^3 + A_5K_3^2$$

$$H_{10} = (1 + A_7K_3 + A_9K_3^2)\ K_1^2$$

$$H_{11} = (A_6K_3 + A_8K_3^2)\ K_1^2$$

$$H_{12} = H_4\ (1 + A_{14}K_3)$$

$$H_{13} = A_{11}K_3^2 + A_{12}K_3 + A_{13}$$

$$H_{14} = A_{10}K_3^2$$

$$H_{15} = (1 + A_{14}K_3)\ K_1^2$$

and

$$K_3 = K_1^2 + K_2^2 \tag{109}$$

After substituting $g = \frac{G_c t_c}{G_f t_f}$ and $f = \frac{N}{G_f t_f}$ into Equations (107) and (108), $w_{mn}$ and $W_{mn}$ are obtained in the following form:

$$w_{mn} = \frac{F H_{16} (G_c + H_{17} G_f)}{\{G_c^2 + H_{18} G_c G_f + H_{19} G_f^2 - (H_{20} G_c + H_{21} G_f) N\}} \tag{110}$$

and

$$W_{mn} = \frac{F H_{37}}{G_c + H_{38} G_f - H_{39} N} \tag{111}$$

where

$$H_{16} = \frac{H_3 H_5 r}{H_9}, \quad H_{17} = \frac{H_6 r}{H_5}, \quad H_{18} = \frac{H_8 r}{H_9}$$

$$H_{19} = \frac{H_7 r^2}{H_9}, \quad H_{20} = \frac{H_{10} r}{H_9 t_f}, \quad H_{21} = \frac{H_{11} r^2}{H_9 t_f}$$

$$H_{37} = \frac{H_{12}}{H_{13}}, \quad H_{38} = \frac{H_{14}}{H_{13}} \quad \text{and} \quad H_{39} = \frac{H_{15}}{H_{13}} \tag{112}$$

The Equations (110) and (111) are for an elastic sandwich plate. According to the correspondence principle discussed in the last section, these two equations for a viscoelastic sandwich plate would take the form:

$$\bar{\bar{w}}_{mn} = \frac{\bar{\bar{F}}\, H_{16}\, (\bar{\bar{G}}_c + H_{17}\bar{\bar{G}}_f)}{\bar{\bar{G}}_c^{\,2} + H_{18}\bar{\bar{G}}_c\bar{\bar{G}}_f + H_{19}\bar{\bar{G}}_f^{\,2} - (H_{20}\bar{\bar{G}}_c + H_{21}\bar{\bar{G}}_f)\bar{\bar{N}}} \tag{113}$$

and

$$\bar{\bar{W}}_{mn} = \frac{\bar{\bar{F}}\, H_{37}}{\bar{\bar{G}}_c + H_{38}\bar{\bar{G}}_f - H_{39}\bar{\bar{N}}} \tag{114}$$

At first, the deflections will be found for F(t) equal to a unit step function and it will be shown that the solution thus obtained could be used for any arbitrarily time dependent load with the help of convolution integrals. Then, for

$F(t) = H(t)$, the Heaviside unit step function,

$$\bar{F} = \frac{1}{s} \tag{115}$$

Also, the axial load is assumed to be:

$$N(t) = N_i\, H(t)$$

$$\bar{N} = \frac{N_i}{s} \tag{116}$$

For the values of F and N given in (115) and (116). Equations (113) and (114) now become:

$$\overline{w}_{mn} = \frac{H_{16}\,(\overline{G}_c + H_{17}\overline{G}_f)}{s^2\{\overline{G}_c^{\,2} + H_{18}\overline{G}_c\overline{G}_f + H_{19}\overline{G}_f^{\,2} - (H_{20}\overline{G}_c + H_{21}\overline{G}_f)\frac{N_i}{s}\}} \qquad (117)$$

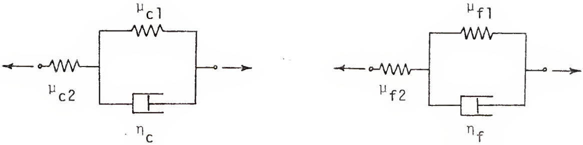
and

$$\overline{W}_{mn} = \frac{H_{37}}{s^2(\overline{G}_c + H_{38}\overline{G}_f - H_{39}\,\frac{N_i}{s}\,)} \qquad (118)$$

## Material Parameters

It is assumed that the core is made of an isotropic and linearly viscoelastic material which can be represented as a 3-parameter model as shown in Figure (4). The upper and lower face layers are assumed to be made of the same material, but different from that of the core. The material which the faces are composed of is also assumed to be isotropic and linearly viscoelastic represented by another 3-parameter model also shown in Figure (4). Each of the two models has a spring in series with a parallel combination of a spring and a dash pot and therefore represents a viscoelastic solid. From here on, the subscripts c and f will refer to the core and the face materials respectively.

From the tabulated chart given in [62], the value of the relaxation modulus in shear for the core material is given by:



(core)

(faces)

Figure 4. Three-parameter models for viscoelastic materials

$$Y_c(t) = \frac{P_{c3}}{P_{c1}} e^{-\frac{t}{P_{c1}}} + P_{c2}\left(1 - e^{-\frac{t}{P_{c1}}}\right) \tag{119}$$

where

$$P_{c1} = \frac{\eta_c}{\mu_{c1} + \mu_{c2}} \quad ; \quad P_{c2} = \frac{\mu_{c1}\mu_{c2}}{\mu_{c1} + \mu_{c2}}$$

and

$$P_{c3} = \frac{\eta_c \mu_{c2}}{\mu_{c1} + \mu_{c2}} \tag{120}$$

From (119) and (120) it follows that for the core,

$$Y_c = \frac{P_{c6}(s + P_{c5})}{s(s + P_{c4})} \tag{121}$$

where

$$P_{c4} = \frac{1}{P_{c1}}, \quad P_{c5} = \frac{P_{c2}}{P_{c3}}, \quad P_{c6} = P_{c3}P_{c4}$$

In a similar way, the expression for the relaxation modulus of the face material can be obtained as follows:

$$Y_f(t) = \frac{P_{f3}}{P_{f1}} e^{-\frac{t}{P_{f1}}} + P_{f2} \left(1 - e^{-\frac{t}{P_{f1}}}\right) \tag{122}$$

and therefore,

$$\overline{Y}_f = \frac{P_{f6} (s + P_{f5})}{s(s + P_{f4})} \tag{123}$$

in which the quantities $P_{f1}$ through $P_{f6}$ are defined in a way exactly similar to that for the core.

Since in the present analysis the external loads applied to the plate are Heaviside unit step functions of time, the quantities $G_c$ and $G_f$ in Equations (117) and (118) are the same as the quantities $Y_c$ and $Y_f$ respectively in Equations (119) and (122).

## Antisymmetric Deflection

In Equation (117), the quantities $\overline{Y}_c$ and $\overline{Y}_f$ whose values are given in (119) and (123) are substituted for

$\bar{G}_c$ and $\bar{G}_f$ respectively, and the result is:

$$\bar{w}_{mn} = \frac{H_{30}(s^2 + H_{31}s + H_{32})(s+P_{c4})(s+P_{f4})}{s(s^4 + H_{33}s^3 + H_{34}s^2 + H_{35}s + H_{36})} \tag{124}$$

where

$$H_{30} = \frac{H_{16}H_{32}}{H_{25}}, \qquad H_{31} = \frac{H_{23}}{H_{22}}, \qquad H_{32} = \frac{H_{24}}{H_{22}},$$

$$H_{33} = \frac{H_{26}}{H_{25}}, \quad H_{34} = \frac{H_{27}}{H_{25}}, \quad H_{35} = \frac{H_{28}}{H_{25}}, \quad H_{36} = \frac{H_{29}}{H_{25}}$$

$$H_{22} = (P_{c6} + H_{17}P_{f6})$$

$$H_{23} = P_{c6}(P_{c5} + P_{f4}) + H_{17}\, P_{f6}(P_{f5} + P_{c4})$$

$$H_{24} = P_{c6}P_{c5}P_{f4} + H_{17}P_{f6}P_{f5}P_{c4}$$

$$H_{25} = P_{c6}^{\;2} + H_{18}P_{c6}P_{f6} + H_{19}P_{f6}^{\;2} - (H_{20}P_{c6} + H_{21}P_{f6})N_i$$

$$H_{26} = 2P_{c6}^2\,(P_{c5} + P_{f4}) + H_{18}P_{c6}P_{f6}(P_{c4}+P_{f4}+P_{c5}+P_{f5}) + 2H_{19}P_{f6}^2(P_{c4}+P_{f5})$$
$$-\{H_{20}P_{c6}(2P_{f4}+P_{c5}+P_{c4}) + H_{21}P_{f6}(2P_{c4}+P_{f5}+P_{f4})\}\, N_i$$

$$H_{27} = P_{c6}^2(P_{c5}^2+P_{f4}^2+4P_{c5}P_{f4}) + H_{18}P_{c6}P_{f6}\{P_{c5}P_{f5}+P_{c4}P_{f4}+(P_{c5}+P_{f5})(P_{c4}+P_{f4})\}$$

(contd.)

$$+H_{19}P_{f6}^{2}(P_{f5}^{2}+P_{c4}^{2}+4P_{f5}P_{c4}) - [H_{20}P_{c6}\{P_{c5}P_{c4}+P_{f4}^{2}(P_{c5}+P_{c4})\}$$

$$+H_{21}P_{f6}\{P_{f5}P_{f4} + P_{c4}^{2} + 2P_{c4}(P_{f5} + P_{f4})\}]\ N_{i}$$

$$H_{28} = 2P_{c6}^{2}(P_{c5}P_{f4}^{2}+P_{f4}P_{c5}^{2}) + H_{18}P_{c6}P_{f6}\{(P_{c5}+P_{f5})P_{c4}P_{f4}+(P_{c4}+P_{f4})P_{c5}P_{f5}\}$$

$$+2H_{19}P_{f6}^{2}(P_{f5}P_{c4}^{2}+P_{c4}P_{f5}^{2}) - [H_{20}P_{c6}\{(P_{c5}+P_{c4})P_{f4}^{2}+2P_{f4}P_{c5}P_{c4}\}$$

$$+H_{21}P_{f6}\{(P_{f5}+P_{f4})P_{c4}^{2} + 2P_{c4}P_{f5}P_{f4}\}]\ N_{i}$$

$$H_{29} = P_{c6}^{2}P_{c5}^{2}P_{f4}^{2} + H_{18}P_{c6}P_{f6}P_{c5}P_{f5}P_{c4}P_{f4} + H_{19}P_{f6}^{2}P_{f5}^{2}P_{c4}^{2}$$

$$-(H_{20}P_{c6}P_{c5}P_{c4}P_{f4}^{2} + H_{21}P_{f6}P_{f5}P_{f4}P_{c4}^{2})\ N_{i} \tag{125}$$

The next task is to find the inverse transform of the expression in (124). As the first step to achieve this, the right hand side of (124) is written as follows by the method of partial fractions:

$$\overline{w}_{mn} = H_{30}\left[\frac{B_1}{s} + \frac{B_2 s^3 + B_3 s^2 + B_4 s + B_5}{s^4 + H_{58}s^3 + H_{59}s^2 + H_{60}s + H_{61}}\right]$$

$$= H_{30}\left[\frac{B_1}{s} + \frac{Q(s)}{P(s)}\right] \tag{126}$$

where

$$Q(s) = B_2 s^3 + B_3 s^2 + B_4 s + B_5$$

$$P(s) = s^4 + H_{58} s^3 + H_{59} s^2 + H_{60} s + H_{61} \qquad (127)$$

and the quantities $B_1$, $B_2$, $B_3$, $B_4$ and $B_5$ can be easily found using the method of partial fractions.

Let $b_1$, $b_2$, $b_3$ and $b_4$ be the roots of the polynomial P(s), i.e.,

$$P(s) = (s - b_1)(s - b_2)(s - b_3)(s - b_4) \qquad (128)$$

The inverse transform of $w_{mn}$ can have different expressions depending on the values of the roots $b_1$, $b_2$, $b_3$ and $b_4$. All these different possibilities will now be explored.

Case (I): All Real Roots

Within this category, there can be several possibilities as follows:

(a) All roots different from each other

i.e., $b_1 \neq b_2 \neq b_3 \neq b_4$

From the table of Laplace transforms [63] the inverse transform is obtained and the value of $w_{mn}$ is given by

$$w_{mn}(t) = H_{30}\,[B_1 + \sum_{k=1}^{4} \frac{Q(b_k)}{P_k(b_k)}\, e^{b_k t}]$$

where

$$P_k(s) = \frac{P(s)}{s-b_k} \tag{129}$$

(b) Two equal roots

Let $b_1 = b_2 \neq b_3 \neq b_4$

Let

$$\frac{Q(s)}{P(s)} = \frac{B_6 s + B_7}{(s - b_1)^2} + \frac{B_8 s + B_9}{(s-b_3)(s-b_4)}$$

in which $B_6$, $B_7$, $B_8$ and $B_9$ can be found easily. Substituting the above quantity into (126) and referring to [26], the following result is obtained:

$$w_{mn}(t) = H_{30}[B_1 + B_6 \{1 + (B_{10} + b_1)t\} e^{b_1 t}$$

$$+ B_8 \{\frac{b_3+B_{11}}{b_3-b_4} e^{b_3 t} + \frac{b_4+B_{11}}{b_4-b_3} e^{b_4 t}\}] \tag{130}$$

where

$$B_{10} = \frac{B_7}{B_6} \text{ and } B_{11} = \frac{B_9}{B_8}$$

(c) Three equal roots

Let $b_1 = b_2 = b_3 \neq b_4$

Then from (126),

$$w_{mn}(t) = H_{30}[B_1 + B_2 \{\frac{b_4^3 + B_{12}b_4^2 + B_{13}b_4 + B_{14}}{(b_4 - b_1)^3} e^{b_4 t}$$

$$+ \frac{3b_4b_1^2 - 3b_4^2b_1 - b_1^3 - B_{12}b_4^2 - B_{13} + B_{14}}{(b_4 - b_1)^3} e^{b_1 t}$$

$$+ \frac{(B_{12}-3b_4)b_1^2 + 2b_1^3 - 2B_{12}b_4b_1 - B_{13}b_4 - B_{14}}{(b_1 - b_4)^2} te^{b_1 t}$$

$$+ \frac{B_{12}b_1^2 + b_1^3 + B_{13}b_1 + B_{14}}{2(b_1 - b_4)} t^2 e^{b_1 t}\}]$$

where

$$B_{12} = \frac{B_3}{B_2}, \quad B_{13} = \frac{B_4}{B_2}, \quad B_{14} = \frac{B_5}{B_2} \tag{131}$$

(d) All equal roots

Here $b_1 = b_2 = b_3 = b_4$.

Then the inverse of (126) is given by:

$$w_{mn}(t) = H_{30}[B_1 + B_2 \{1 + (B_{12} + 3b_1)t + \frac{B_{13}+2B_{12}b_1+3b_1^2}{2} t^2$$

$$+ \frac{B_{14} + B_{13}b_1 + B_{12}b_1^2 + b_1^3}{6} t^3\} e^{b_1 t}] \tag{132}$$

where the values of $B_{12}$, $B_{13}$ and $B_{14}$ are given in (131).

(e) Two pairs of equal roots

In this case, $b_1 = b_2 \neq b_3 = b_4$

This can be treated as a special case of all complex roots, to be treated next, in which the complex parts of the roots are zeros.

## Case (II): All Complex Roots

Since the complex roots occur in pairs, let

$$b_1 = \alpha_1 + i\beta_1$$

$$b_2 = \alpha_1 - i\beta_1$$

$$b_3 = \alpha_2 + i\beta_2$$

$$b_4 = \alpha_2 - i\beta_2$$

Then,

$$(s - b_1)(s - b_2) = (s - \alpha_1)^2 + \beta_1^2$$

and

$$(s - b_3)(s - b_4) = (s - \alpha_2)^2 + \beta_2^2$$

Hence,

$$P(s) = \{(s - \alpha_1)^2 + \beta_1^2\}\{(s - \alpha_2)^2 + \beta_2^2\}$$

Now, let

$$\frac{Q(s)}{P(s)} = \frac{D_1 s + D_2}{(s-\alpha_1)^2 + \beta_1^2} + \frac{D_3 s + D_4}{(s-\alpha_2)^2 + \beta_2^2} \qquad (133)$$

in which the quantities $D_1$, $D_2$, $D_3$ and $D_4$ can be solved by the method of partial fractions.

Substituting (133) into (126) and looking up standard tables of Laplace transforms like [63], the result obtained is:

$$w_{mn}(t) = H_{30}\,[B_1 + D_1 e^{\alpha_1 t}\cos\beta_1 t + \frac{D_1\alpha_1 + D_2}{\beta_1} e^{\alpha_1 t}\sin\beta_1 t + D_3 e^{\alpha_2 t}\cos\beta_2 t$$

$$+ \frac{D_3\alpha_2 + D_4}{\beta_2} e^{\alpha_2 t}\sin\beta_2 t] \qquad (134)$$

For the case of two pairs of real and equal roots, as mentioned earlier, the results given in (134) can be used by taking $\beta_1 = \beta_2 = 0$, and to avoid division by zero during computation, the quantities $\frac{\mathrm{Sin}\beta_1 t}{\beta_1}$ and $\frac{\mathrm{Sin}\beta_2 t}{\beta_2}$ can be written as $(\frac{\mathrm{Sin}\beta_1 t}{\beta_1 t})t$ and $(\frac{\mathrm{Sin}\beta_2 t}{\beta_2 t})t$ respectively, and these two quantities then become equal to t as $\beta_1 \to 0$ and $\beta_2 \to 0$.

In a similar way, the results given in (134) can also be used in the case where there are a pair of complex roots and a pair of real and equal roots.

Case III: Two Real and Unequal Roots and a Pair of Complex Roots

Let $b_1 = \gamma + i\delta$

and

$$b_2 = \gamma - i\delta$$

and

$$b_3 \neq b_4 \quad \text{where } b_3 \text{ and } b_4 \text{ are real roots.}$$

Then, $(s - b_1)(s - b_2) = (s - \gamma)^2 + \delta^2$.

Let

$$\frac{Q(s)}{P(s)} = \frac{D_5 s + D_6}{(s-\gamma)^2+\delta^2} + \frac{D_7 s + D_8}{(s-b_3)(s-b_4)} \tag{135}$$

where the quantities $D_5$, $D_6$, $D_7$ and $D_8$ can be readily found by using the method of partial fractions.

Substituting in (126) and upon performing the inverse transform, the following result is obtained:

$$w_{mn}(t) = H_{30}\,[B_1 + D_5 e^{\gamma t} \operatorname{Cos} \delta t + \frac{D_5\gamma + D_6}{\delta} e^{\gamma t} \operatorname{Sin} \delta t$$

$$+ D_7\left( \frac{b_3 + D_9}{b_3 - b_4} e^{b_3 t} + \frac{b_4 + D_9}{b_4 - b_3} e^{b_4 t}\right)] \tag{136}$$

where

$$D_9 = \frac{D_8}{D_7}$$

## Case (IV): Two Real and Equal Roots and a Pair of Complex Roots

This is a special case of (II) where either $\beta_1$ or $\beta_2$ is zero.

After $w_{mn}(t)$ has been found in any of the above mentioned ways, it is to be substituted into the series expression given in (89) to compute the antisymmetric deflection w(x, y, t) for any value of x, y and t.

Symmetric Deflection

In Equation (118), the quantities $\overline{G}_c$ and $\overline{G}_f$ are replaced by $\overline{Y}_c$ and $\overline{Y}_f$ respectively and the values of $\overline{Y}_c$ and $\overline{Y}_f$ as given in (121) and (123) are substituted. The following result is obtained:

$$\overline{W}_{mn} = \frac{H_{45}(s^2 + H_{43}s + H_{44})}{s(s^2 + H_{46}s + H_{47})} \tag{137}$$

where

$$H_{43} = P_{c4} + P_{f4}, \qquad H_{44} = P_{c4}\,P_{f4}$$

$$H_{45} = \frac{H_{37}}{H_{40}} \qquad H_{46} = \frac{H_{41}}{H_{40}} \qquad H_{47} = \frac{H_{42}}{H_{40}}$$

$$H_{40} = P_{c6} + H_{38}P_{f6} - H_{39}\,N_i$$

$$H_{41} = P_{c6}(P_{c5}+P_{f4}) + H_{38}P_{f6}(P_{f5}+P_{c4}) - H_{39}(P_{c4} + P_{f4})N_i$$

and

$$H_{42} = P_{c6}P_{c5}P_{f4} + H_{38}P_{f6}P_{f5}P_{c4} - H_{39}P_{c4}P_{f4}N_i \tag{138}$$

Let $(s^2 + H_{46}s + H_{47}) = (s + \alpha)(s + \beta)$.

The following three cases may arise depending on the values of $\alpha$ and $\beta$:

Case (I): Real and Unequal Roots

Here, $\alpha \neq \beta$; $\alpha$ and $\beta$ are real.

Then taking the inverse transform of (137), it is found that

$$W_{mn}(t) = H_{45}\left[ \frac{H_{44}}{H_{47}} + \frac{\alpha^2 - H_{43}\alpha + H_{44}}{\alpha(\alpha - \beta)} e^{-\alpha t} - \frac{\beta^2 - H_{43}\beta + H_{44}}{\beta(\alpha - \beta)} e^{-\beta t}\right] \quad (139)$$

Case (II): Real and Equal Roots

This can be treated as a special case of (III), to be considered next.

Case (III): Complex Roots

Let $\alpha = \gamma_1 + i\delta_1$

$\alpha = \gamma_1 - i\delta_1$

Then the inverse transform of (137) is:

$$W_{mn}(t) = H_{45}\left[ \frac{H_{44}}{H_{47}} + D_{10}e^{-\gamma_1 t} \operatorname{Cos} \delta_1 t + \frac{D_{11} - D_{10}\gamma_1}{\delta_1} e^{-\gamma_1 t} \operatorname{Sin} \delta_1 t\right] \quad (140)$$

where

$$D_{10} = 1 - \frac{H_{44}}{H_{47}}$$

and

$$D_{11} = H_{43} - \frac{H_{44}H_{46}}{H_{47}}$$

The result (140) can be used for the case where the roots are real and equal by putting $\delta_1 = 0$. Division by zero can be avoided as shown in connection with the antisymmetric deflection.

The value of $W_{mn}(t)$ so obtained can now be substituted into the expression given in (89) to obtain the symmetric deflection $W(x, y, t)$ for any x, y and t.

An important remark can be made at this point. In whichever of the above mentioned ways $w_{mn}(t)$ or $W_{mn}(t)$ are calculated, it is always found to be consisting of two parts, a constant term plus a few exponential terms. For $N_i$ less than its critical value for either mode, antisymmetric or symmetric, for any given t (it will be shown in the next chapter that the critical value of the axial stress resultant N is a function of time), both $w_{mn}(t)$ and $W_{mn}(t)$ must be finite, and this requires that the real parts of the roots of P(s) and $(s^2 + H_{46}s + H_{47})$ be negative. It is easy to prove that this requirement is satisfied for $(s^2 + H_{46}s + H_{47})$. It is seen that in P(s), all the coefficients are of the same sign which is a necessary but not a sufficient condition for the real parts of all the roots to be negative. To see that both the necessary and sufficient conditions are satisfied, one should check the Routh-Hurwitz criterion which is given in any standard text on the stability of dynamic systems like [64]. However, it can be shown numerically (numerical results are given in the last chapter) that this condition is indeed satisfied, i.e., the time dependent terms in the expressions of $w_{mn}(t)$ and $W_{mn}(t)$ are exponentially decreasing functions of time. A few more remarks have been made in the discussion chapter.

Thus,

$$w_{mn}\ (\text{at } t = \infty) = H_{30}B_1$$

and

$$W_{mn}\ (\text{at } t = \infty) = \frac{H_{45}H_{44}}{H_{47}} \tag{141}$$

## Stability Analysis

If the transverse loads $p_1$ and $p_2$ are zero, it is found from (104), (106), (109) and (112) that the quantities $H_{16}$ and $H_{37}$ are also equal to zero. Then from (113) and (114) it follows that the deflections $w_{mn}$ and $W_{mn}$ of the elastic sandwich plate can grow indefinitely if N takes up the following two values corresponding to the antisymmetric instability and the symmetric instability respectively:

$$N_{cr(a)} = \frac{G_c^{\,2} + H_{18}G_cG_f + H_{19}G_f^{\,2}}{H_{20}G_c + H_{21}G_f} \qquad (142)$$

(antisymmetric)

and

$$N_{cr(s)} = \frac{G_c + H_{38}G_f}{H_{39}} \qquad (143)$$

(symmetric)

where the quantities $N_{cr(a)}$ and $N_{cr(s)}$ denote the critical values of N for antisymmetric and symmetric buckling respectively.

According to the method used earlier, the time dependent critical values of N for a viscoelastic sandwich plate are obtained from (142) and (143) as follows:

$$\bar{\bar{N}}_{cr(a)} = \frac{\bar{\bar{G}}_c^{\,2} + H_{18}\bar{\bar{G}}_c\bar{\bar{G}}_f + H_{19}\bar{\bar{G}}_f^{\,2}}{H_{20}\bar{\bar{G}}_c + H_{21}\bar{\bar{G}}_f}$$

or,

$$\bar{N}_{cr(a)} = \frac{\bar{G}_c + H_{18}\bar{G}_c\bar{G}_f + H_{18}\bar{G}_f^{\,2}}{H_{20}\bar{G}_c + H_{21}\bar{G}_f} \qquad (144)$$

and

$$\bar{\bar{N}}_{cr(s)} = \frac{\bar{\bar{G}}_c + H_{38}\bar{\bar{G}}_f}{H_{39}}$$

or

$$\bar{N}_{cr(s)} = \frac{\bar{G}_c + H_{38}\bar{G}_f}{H_{39}} \qquad (145)$$

Replacing $\overline{G}_c$ and $\overline{G}_f$ by $\overline{Y}_c$ and $\overline{Y}_f$ respectively in Equations (144) and (145), and substituting (121) and (123) for $\overline{Y}_c$ and $\overline{Y}_f$, the following results are obtained:

$$\overline{N}_{cr(a)} = \frac{H_{56}(s^4 + H_{57}s^3 + H_{58}s^2 + H_{59}s + H_{60})}{s(s = P_{c4})(s + P_{f4})(s^2 + H_{61}s + H_{61})} \qquad (146)$$

and

$$\overline{N}_{cr(s)} = \frac{H_{66}(s^2 + H_{67}s + H_{68})}{s(s + P_{c4})(s + P_{f4})} \qquad (147)$$

where

$$H_{56} = \frac{H_{48}}{H_{53}}, \quad H_{57} = \frac{H_{49}}{H_{48}}, \quad H_{58} = \frac{H_{50}}{H_{48}}, \quad H_{59} = \frac{H_{51}}{H_{48}}$$

$$H_{60} = \frac{H_{52}}{H_{48}}, \quad H_{61} = \frac{H_{54}}{H_{53}}, \quad H_{62} = \frac{H_{55}}{H_{53}},$$

The quantities $H_{48}$ through $H_{52}$ are equal to the expressions for the quantities $H_{25}$ through $H_{29}$ respectively given in Equations (125) if $N_i$ is set equal to zero.

$$H_{53} = H_{20}P_{c6} + H_{21}P_{f6}$$

$$H_{54} = H_{20}P_{c6}(P_{c5} + P_{f4}) + H_{21}P_{f6}(P_{f5} + P_{c4})$$

$$H_{55} = H_{20}P_{c6}P_{c5}P_{f4} + H_{46}P_{f6}P_{f5}P_{c4}$$

$$H_{66} = \frac{H_{63}}{H_{39}}, \quad H_{67} = \frac{H_{64}}{H_{53}}, \quad H_{68} = \frac{H_{65}}{H_{63}},$$

$$H_{63} = P_{c6} + H_{38}P_{f6}$$

$$H_{64} = P_{c6}(P_{c5} + P_{f4}) + H_{38}P_{f6}(P_{f5} + P_{c4})$$

and

$$H_{65} = P_{c6}P_{c5}P_{f4} + H_{38}P_{f6}P_{f5}P_{c4} \tag{148}$$

The inverse transforms of (146) and (147) have to be found, as will be shown below.

Antisymmetric Buckling

Equation (146) is written as follows:

$$N_{cr(a)} = H_{56}\left[ \frac{A}{s} + \frac{Bs + C}{(s+P_{c4})(s+P_{f4})} + \frac{Ds + E}{s^2 + H_{61}s + H_{62}} \right] \tag{149}$$

in which the coefficients A, B, C, D and E can be found easily. The inverse transforms of the three terms within the brackets will be found separately.

(I) Inverse transform of $\frac{A}{s} = A$ (150)

(II) Inverse of transform of $\frac{Bs + C}{(s+P_{c4})(s+P_{f4})}$

$$= B \left[\frac{P_{c4} - (\frac{C}{B})}{P_{c4} - P_{f4}} e^{-P_{c4}t} + \frac{P_{f4} - (\frac{C}{B})}{P_{f4} - P_{c4}} e^{-P_{f4}t}\right] \; . \; . \; . \text{ if } P_{c4} \neq P_{f4}$$

$$= B \left[1 + (\frac{C}{B} - P_{c4})t\right] e^{-P_{c4}t} \; . \; . \; . \text{ if } P_{c4} = P_{f4} \tag{151}$$

(III) Let $s^2 + H_{61}s + H_{62} = (s + \phi_1)(s + \phi_2)$

(a) If $\phi_1$ and $\phi_2$ are real, finding the inverse of the third term within the brackets in (149) becomes similar to that for the second term in the same expression.

(b) If $\phi_1$ and $\phi_2$ are complex, let

$\phi_1 = \theta_1 + i\theta_2$ , and

$\phi_2 = \theta_1 - i\theta_2$

Then the inverse transform of the third term

$$= D\,[e^{-\theta_1 t} \operatorname{Cos} \theta_2 t + \frac{(\frac{E}{D}) - \theta_1}{\theta_2} e^{-\theta_1 t} \operatorname{Sin} \theta_2 t] \qquad (152)$$

Substituting the above expressions for the inverse transforms in (149), the value of $N_{cr(a)}(m, n, t)$ is now obtained.

## Symmetric Buckling

From Equation (147), the following results are obtained:

$$N_{cr(s)}(m,n,t) = H_{66}\,[\frac{H_{68}}{P_{c4}P_{f4}} + \frac{P_{c4}^2 - H_{67}P_{c4} + H_{68}}{P_{c4}(P_{c4} - P_{f4})} e^{-P_{c4}t}$$

$$- \frac{P_{f4}^2 - H_{67}P_{f4} + H_{68}}{P_{f4}(P_{c4} - P_{f4})} e^{-P_{f4}t}] \qquad \text{for } P_{c4} \neq P_{f4}$$

and

$$N_{cr(s)}(m,n,t) = H_{66}\left[\frac{H_{68}}{P_{c4}^{2}} - \left\{\frac{H_{68}}{P_{c4}^{2}} - 1 + \left(P_{c4} - H_{67} + \frac{H_{68}}{P_{c4}}\right) t\right\} e^{-P_{c4}t}\right]$$

$$\text{for } P_{c4} = P_{f4} \qquad (153)$$

It can be easily shown that the time dependent parts in the expression of both $N_{cr(a)}$ and $N_{cr(s)}$ are exponentially decreasing functions of time. Then at $t = \infty$,

$$N_{cr(a)}(\text{at } t = \infty) = H_{56}\ A$$

and

$$N_{cr(s)}(\text{at } t = \infty) = \frac{H_{66}H_{68}}{P_{c4}P_{f4}} \qquad (154)$$

Numerical results have been given in the last chapter, where the results obtained in this section have been discussed.

## Elastic Faces and Viscoelastic Core

The analysis of a sandwich plate with elastic faces and viscoelastic core comes out as a special case of the analysis presented in this chapter. In the three-parameter viscoelastic model of the face layers shown in Figure (4), if either $\mu_{f1}$ or $\eta_f$ is set equal to infinity, or if $\eta_f$ if set equal to zero, the model is reduced to a single spring of stiffness equal to $\mu_f2$ or two springs of stiffness $\mu_{f1}$ and $\mu_{f2}$ in series, either of which represents an elastic material. However, it is found to be convenient if $\eta_f$ is set equal to infinity which means that $P_{f1} = \infty$ and $P_{f3} = \infty$, and therefore $P_{f4} = 0$ and $P_{f5} = 0$. But setting $P_{f4} = 0$ and $P_{f5} = 0$ in the final expressions of $w_{mn}$, $W_{mn}$, $N_{cr(a)}$ and $N_{cr(s)}$ would end up with expressions like zero divided by zero. To avoid this difficulty, one should set $P_{f4} = 0$ and $P_{f5} = 0$ in (124), (137), (146) and (147) and calculate the quantities $w_{mn}$, $W_{mn}$, $N_{cr(a)}$ and $N_{cr(s)}$ respectively following a procedure exactly similar to the one shown before.

## Response to a Load which Is an Arbitrary Function of Time

In the analysis of viscoelastic sandwich plate presented so far, it has been assumed that the transverse loads p' and p" are step functions in time, and the values of $w_{mn(t)}$ and $W_{mn(t)}$ obtained are therefore corresponding to step loads. But these results can be generalized for loads which are arbitrary functions of time as follows:

Let the transverse loads p' and p" be uniform throughout the plate. Then, from (106),

$$H_3 = \frac{16H_1}{\pi^2 mn} \quad \text{and} \quad H_4 = \frac{16H_2}{\pi^2 mn} \; ; \qquad \begin{matrix} m = 1, 3, 5, \ldots\ldots\infty \\ n = 1, 3, 5, \ldots\ldots\infty \end{matrix}$$

Thus, for unit loads $H_1$, $H_2$, $H_3$, and $H_4$ become

$H_{3(U)} = \frac{16}{\pi^2 mn}$ and $H_{4(U)} = \frac{16}{\pi^2 mn}$, (U) signifies unit load

and let the corresponding deflections which are proportional to $H_3$ and $H_4$ become $w_{mn(U)}$ and $W_{mn(U)}$. Let

$$\underline{p}' = p_1(x,y)f(t)$$

and

$$\underline{p}'' = p_2(x,y)f(t)$$

where

$$f(t) = 0 \text{ for } t \leq 0$$

$$= \text{an arbitrary function of time for } t > 0$$

Then it follows that the deflections $w_{mn(ar)}$ and $W_{mn(ar)}$ for loads which are arbitrary functions [f(t)] of time are given in the form of convolution integrals as follows:

$$w_{mn(ar)}(t) = \int_{o}^{t} w_{mn(U)}(t - \tau) \frac{df}{d\tau} d\tau \quad (155)$$

and

$$W_{mn(ar)}(t) = \int_{o}^{t} W_{mn(U)}(t - \tau) \frac{df}{d\tau} d\tau \quad (156)$$

where $\tau$ = time variable and the subscript (ar) denotes arbitrary function of time.

An important remark can be made at this point. Unlike the transverse loads, the axial load N appears nonlinearly in the expressions of $w_{mn}(t)$ and $W_{mn}(t)$; therefore generalization similar to the above cannot be made for N.

## Special Cases

The case of elastic faces and viscoelastic core has been discussed already. The deflections of a sandwich plate under transverse loads only can be readily obtained from the expressions of w(t) and W(t) by setting N = 0. Simpler models (like the two-parameter Maxwell or Kelvin-Voigt models) for the viscoelastic materials can be treated as special cases. Finally, the elastic sandwich plate problem itself can be obtained as a special case of the present analysis by modifying the values of $\mu_{c1}$, $\mu_{c2}$, $\eta_c$, $\mu_{f1}$, $\mu_{f2}$ and $\eta_f$.

## Comparison with Other Theories

As mentioned in the first chapter, very few reports on the study of viscoelastic sandwich plates are available. Malmstrong's work [44] considers time dependent creep deformation of circular sandwich plates, but does not consider instability. Also the deformation is due to creep beyond the yield point and in the plastic range, and therefore is quite different from the present analysis except that both analyses have considered time dependent quasi-static deflections.

The work by Chang and Fang [43] is closer to the present analysis. They have considered the dynamic effects for a transverse step load and then for an arbitrary transverse load. However, they have neglected the viso-elastic effects of the faces, the compressibility of the core and the bending rigidities of the faces in addition to the assumption that faces do not take up any shearing load. None of these assumptions have been made in the present analysis. Also, they have not considered the problem of instability. Therefore, excepting the considerations of dynamic effects, the present analysis can be considered to be more general than [43].

The work that comes closest to the present analysis is the one by Lisy [47], who has treated both the problems of bending and stability of a viscoelastic sandwich plate. The constitutive equations for the viscoelastic material considered in [47] are the same as in the present analysis. Also, the form of solution for w(t) is the same in both. However, as mentioned earlier in Chapter 1, Lisy has made many assumptions which have not been made in the present analysis. Also Lisy has calculated the critical load only at $t = \infty$, while the present analysis gives expressions for the critical loads for all $t \geq 0$. Additionally, he has considered only antisymmetric deflection and buckling while the present analysis considers deflection and buckling for both the antisymmetric and symmetric modes. The present analysis is therefore much more general than the one by Lisy.

# CHAPTER 5

# RESULTS, DISCUSSION AND CONCLUSIONS

## Results

The numerical values of w(t) and W(t), which are the time dependent antisymmetric and symmetric deflections respectively of a viscoelastic sandwich plate considered in the last chapter, have been calculated and reported in this chapter. The values of $N_{cr(a)}(t)$ and $N_{cr(s)}(t)$, which are the time dependent critical values of the axial load for antisymmetric and symmetric modes of buckling respectively, have also been calculated.

Since the present analysis aims at studying the viscoelastic behavior of a sandwich plate, each of the viscoelastic parameters (except $c_8$) has been considered over a range so that the changes in the response of the plate for the corresponding changes in these parameters can be determined, and the other geometrical parameters have been given some specific values which have been kept unchanged throughout all the calculations. All the quantities which appear in the calculations and results are in non-dimensional forms. The following non-dimensional quantities are defined as:

$c_1 = b/a$ , $c_2 = {}^{t}c/a$ , $c_3 = \mu_{c2}/\mu_{c1}$ ,

$c_4 = \mu_{f1}/\mu_{c1}$ , $c_5 = \mu f_2/\mu_{c1}$ , $c_6 = (p_1 + p_2)/\mu_{c1}$

$c_7 = (p_1 - p_2)/\mu_{c1}$ , $c_8 = \eta_f/\eta_c$ ,

$c_t = t/\tau_c$ (where $\tau_c = \eta_c/\mu_{c1}$)

$c_N = N_i/(\mu_{c1}a)$, $c_x = x/a$ and $c_y = y/b$

In the specific problems considered in this chapter, the following non-dimensional parameters have the values:

$\nu_f = .28$ , $\nu_c = .3$ , $r = 0.067$

$c_1 = 1.0$ , $c_2 = 0.02$ , and $c_8 = 100$

In addition, $c_N = 0.4 \times 10^{-4}$ , $c_6 = 0.5 \times 10^{-3}$ and $c_7 = 0.1 \times 10^{-2}$ for the problems of deflection. Also, all the deflections have been calculated at $c_x = 0.5$ and $c_y = 0.5$, i.e., at the center of the plate. Each of the parameters $c_3$, $c_4$ and $c_5$ has been given different values over a range and both the antisymmetric and symmetric deflections have been calculated for $o \leq t \leq \infty$. Figures 5-9 show these results. From Figure 9, it is found that the symmetric deflection is quite insensitive to changes in the value of $C_5$ over the range from 50 to 1000 for the given values of $C_3$ and $C_4$ shown in the same figure. Almost exactly the same graph as shown in Figure 9 is obtained if the value of $C_5$ is kept fixed at 100, and $C_4$ is allowed to take up any value in the range from 50 to 1000. Thus, it is found that the antisymmetric deflection at any given time is virtually unaffected by any changes in the values of $C_4$ and $C_5$ if these two quantities have relatively high values compared to $C_3$, which is usually the case.

In the buckling problems, it is found that for each of the antisymmetric and symmetric modes, the mode numbers m and n, equal to the numbers of half sine waves in the x and y directions respectively, which give rise to the minimum values of the critical axial load, remain the same for $0 \leq t \leq \infty$. Therefore, the values of m and n for the minimum values of the critical loads can be determined by comparing the different values of the critical loads corresponding to different values of m and n at any value of t between zero and infinity. Here, this has been done at $t = \infty$. Once m and n are determined, the corresponding critical loads are plotted against time for

$0 \leq t \leq \infty$ and shown in Figures 10 and 12.

It can be shown [42] that both $N_{cr(a)}$ and $N_{cr(s)}$ are monotone increasing functions of n; thus, the minimum values of $N_{cr(a)}$ and $N_{cr(s)}$ occur at $n = 1$. But the same is not necessarily true for m. For antisymmetric buckling it is found that for the particular problem considered here, the minimum value of critical axial load occurs either between $m = 1$ and $m = 2$, or at $m = 1$; and in the former case, the values at $m = 1$ and $m = 2$ differ very slightly from each other (the value at $m = 1$ being the smaller one). Therefore $N_{cr(a)}$ has been calculated for $m = 1$ and for $0 \leq t \leq \infty$.

For symmetric buckling, the value of m corresponding to the minimum critical load is very high. This is shown in Figure 11.

In the calculations, the following non-dimensional parameters denoting the critical loads have been used:

$$C_{na} = N_{cr(a)}/(\mu_{c1}a)$$

and

$$C_{ns} = N_{cr(s)}/(\mu_{c1}a)$$

Figures 5 through 12 show the results.

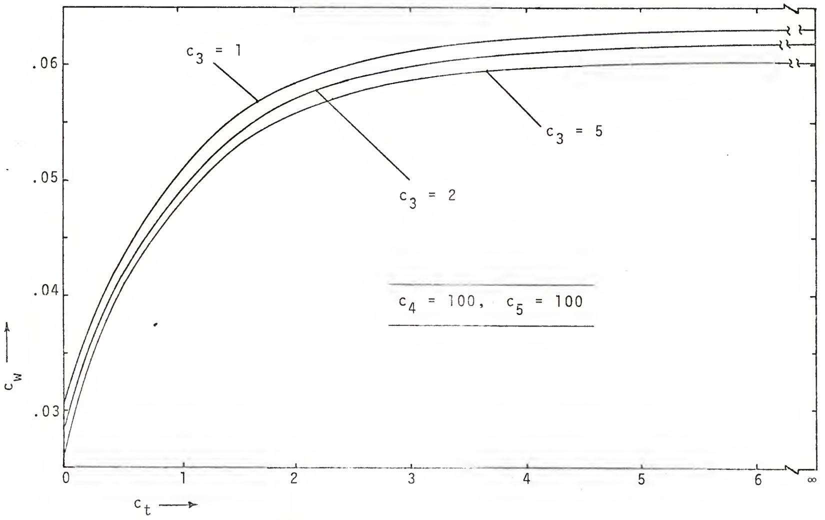


Figure 5. Antisymmetric deflection for different values of $c_3$ vs time

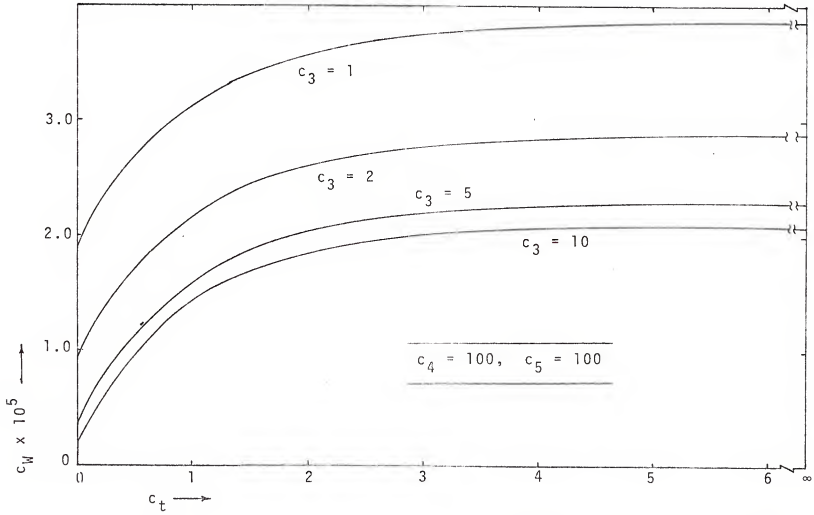


Figure 6. Symmetric deflection for different values of $c_3$ vs time

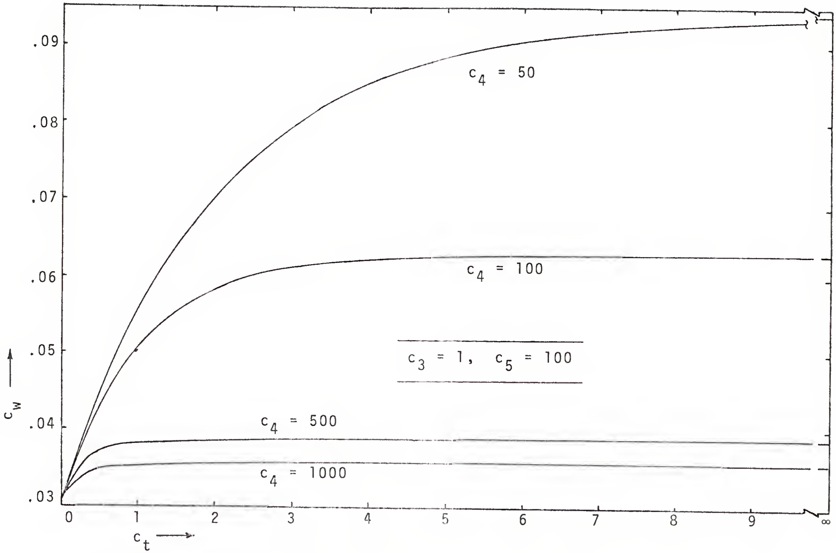


Figure 7. Antisymmetric deflection for different values of $c_4$ vs time

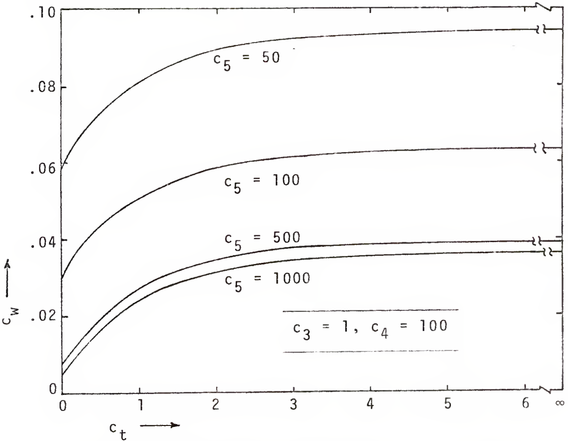


Figure 8. Antisymmetric deflection for different values of $c_5$ vs time

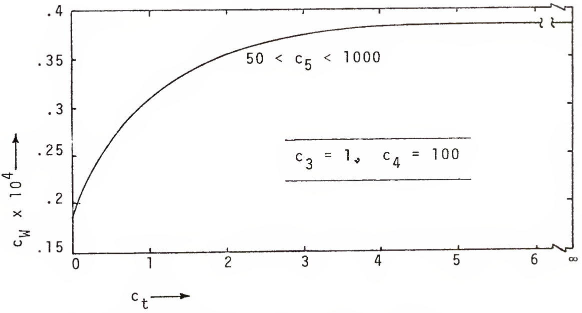


Figure 9. Symmetric deflection for $50 < c_5 < 1000$ vs time

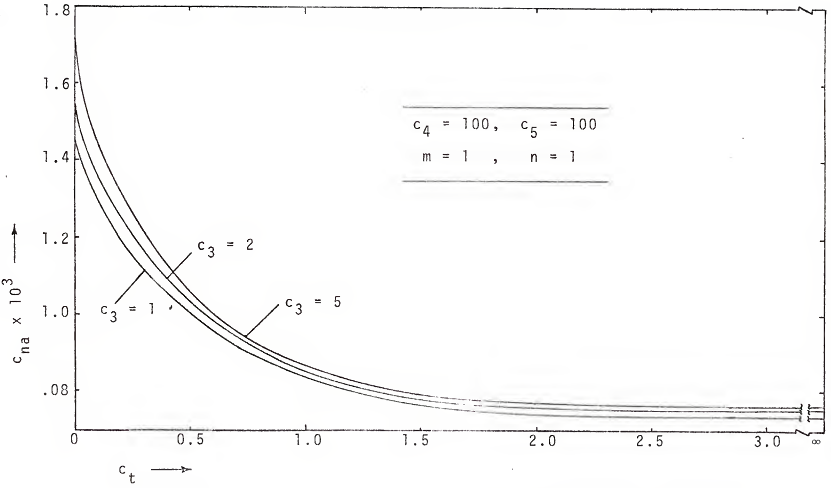


Figure 10. Critical load for antisymmetric buckling vs time

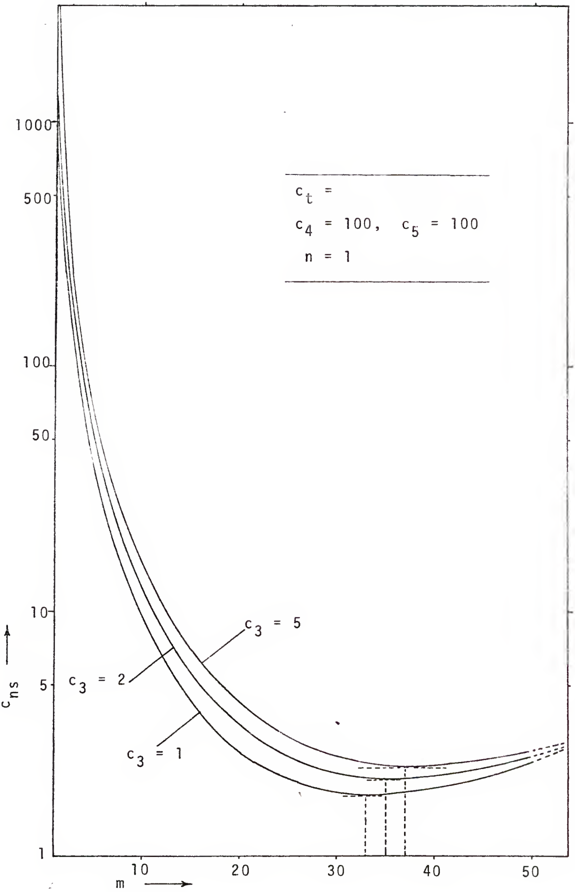


Figure 11. Determination of mode number for symmetric buckling

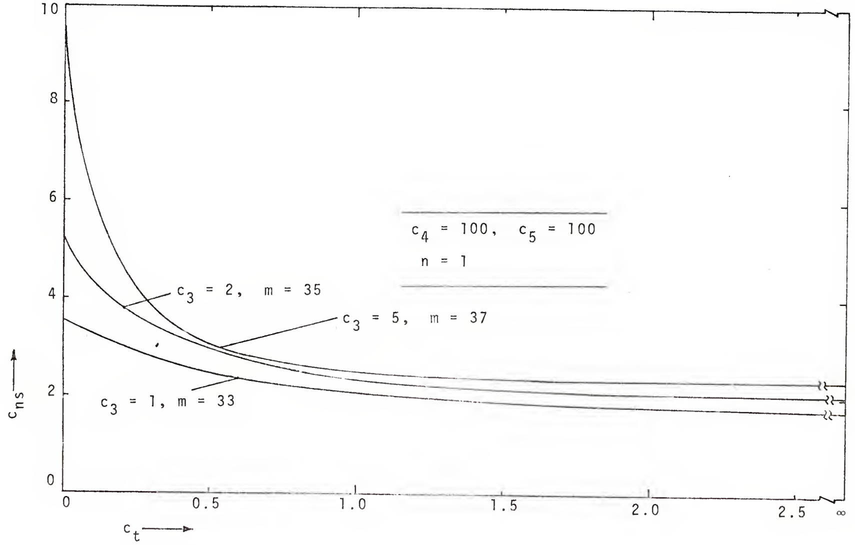


Figure 12. Critical load for symmetric buckling vs time

## Discussion

A viscoelastic material represented by a three-parameter model shown in Figure 4 in a quasi-static problem can be interpreted as an elastic material whose elastic modulus is changing with time, from that of a single spring of stiffness equal to $\mu_{c2}$ (if the material considered is that of the core) at $t = 0$, to that of a combination of two springs of stiffnesses $\mu_{c2}$ and $\mu_{c1}$ in series, at $t = \infty$. At any other time t where $0 \leq t \leq \infty$, the equivalent elastic modulus is that of a combination of springs of stiffnesses $\mu_{c2}$ and $K(t)\mu_{c1}$ in series, the value of the factor K(t) being determined by $\eta_c$, and $K(0) = \infty$, $k(\infty) = 1$. Similar interpretation holds for the material of the faces.

Thus it becomes apparent that the response of the viscoelastic sandwich plate to a given set of loading can be obtained directly from that of an elastic plate in two special cases viz. at $t = 0$ and at $t = \infty$. These results can be used as a check on the results obtained from the analysis of viscoelastic sandwich plates. This is done in the following:

### Symmetric Deflection

For the symmetric deflection, it is readily seen from Equations (139) and (140) of the viscoelastic analysis that

$$W_{mn}(t = 0) = H_{45}$$

and

$$W_{mn}(t = \infty) = \frac{H_{45}H_{44}}{H_{47}} \qquad (157)$$

From (138), (112) and (120) through (123), it is found that $H_{45}$ is equal to the value of $W_{mn}$ as given in Equation (111) of the elastic analysis if $G_c$ and $G_f$ are replaced by $\mu_{c2}$ and $\mu_{f2}$ respectively. Also $(H_{45}H_{44})/H_{47}$ is equal to the value of $W_{mn}$ as given in Equation (111) of the elastic analysis if$G_c$ and $G_f$ are replaced by $K_{ce}$ and $K_{fe}$ respectively, where

$K_{ce} = \frac{\mu_{c2}\,\mu_{c1}}{\mu_{c2}+\mu_{c1}}$ = stiffness of the combination of two springs of stiffnesses $\mu_{c2}$ and $\mu_{c1}$ in series

and

$K_{fe} = \frac{\mu_{f2}\,\mu_{f1}}{\mu_{f2}+\mu_{f1}}$ = stiffness of the combination two springs of stiffnesses $\mu_{f2}$ and $\mu_{f1}$ in series

## Antisymmetric Deflection

As for the antisymmetric deflection, it is found from Equation (141) that

$$w_{mn}(t = \infty) = H_{30}B_1 \tag{158}$$

and from (125) and (120) through (123) it is seen that $H_{30}B_1$ is equal to the value of $w_{mn}$ as given in Equation (110) of the elastic analysis if $G_c$ and $G_f$ are replaced by $K_{ce}$ and $K_{cf}$ respectively.

To find the value of $w_{mn}(t = 0)$, a direct way would be to put t = 0 in one of the Equations (129) through (136). But this involves very long algebraic expressions which are difficult to work with. To get around this difficulty, the following procedure can be followed:

If f(s) = Laplace transform of F(t),

then

$$\underset{s\to\infty}{\mathrm{Lt}}\ s f(s) = F(0) \tag{159}$$

Using (159) and (124) the following is obtained:

$$w_{mn}(t = 0) = \underset{s\to\infty}{\mathrm{Lt}}\ s\bar{w}_{mn}$$

$$= H_{30} \tag{160}$$

It is now readily seen from (125) and (110) that $H_{30}$ is equal to the value of $w_{mn}$ as given in Equation (110) of the elastic analysis if $G_c$ and $G_f$ are replaced by $\mu_{c2}$ and $\mu_{f2}$ respectively.

An independent way of calculating the critical loads $N_{cr(a)}$ and $N_{cr(s)}$ at $t = 0$ and at $t = \infty$ is to consider the expressions $w_{mn}(t = 0)$, $w_{mn}(t = \infty)$, $W_{mn}(t = 0)$ and $W_{mn}(t = \infty)$ as given in (160), (158) and (157). This can serve as a check on the results obtained in the last chapter for $w_{mn}(t)$ and $W_{mn}(t)$ where $0 \leq t \leq \infty$. This is done as follows:

Antisymmetric Buckling

From (160) and (125), it is found that

$$w_{mn}(t = 0) = H_{30} = \frac{H_{16}H_{22}}{H_{25}}$$

The only way in which $w_{mn}(t = 0)$ can become infinity for finite transverse loads is when $H_{25}$ goes to zero, and the condition $H_{25} = 0$ gives, from (125),

$$N_i(t = 0) = N_{cr(a)}(t = 0) = \frac{P_{c6}^2 + H_{18}P_{c6}P_{f6} + H_{19}P_{f6}^2}{H_{20}P_{c6} + H_{21}P_{f6}} \qquad (161)$$

Also,

$$N_{cr(a)}(t = 0) = \underset{s \to \infty}{Lt} \; s\bar{N}_{cr(a)} \qquad \text{[using (159)]}$$

$$= H_{56}$$

$$= \frac{P_{c6}^2 + H_{18}P_{c6}P_{f6} + H_{19}P_{f6}^2}{H_{20}P_{c4} + H_{21}P_{f6}} \qquad (162)$$

From (161) and (162), it is found that the same expression for $N_{cr(a)}(t = 0)$ is obtained in two different ways.

In a similar way, it can be shown that if $N_{cr(a)}$ $(t = \infty)$ is calculated first from $w_{mn}(t = \infty)$ and then from the expression for $N_{cr(a)}(t)$, the same value is obtained as given in (158).

Symmetric Buckling

Each of the values of $N_{cr(s)}(t = 0)$ and $N_{cr}(t = \infty)$ can be obtained in two different ways and can be shown to be the same following a procedure similar to that for antisymmetric buckling.

One important point is to be noted here. The value of the critical load $N_{cr(a)}$ can be obtained from the expression for $w_{mn}(t)$ as shown in this section only at $t = 0$ and $t = \infty$. At any other value of t where $0 < t < \infty$, it can be shown that no definite condition can be obtained for which $w_{mn}(t)$ becomes infinity. So, to obtain the value of

$N_{cr(a)}(t)$ for any time t where $0 < t < \infty$, one must calculate it from Equations (129) through (136).

The same is true of $N_{cr(s)}t$.

## Applicability of the Present Analysis

The present analysis enables one to predict the deflection, antisymmetric or symmetric, at any time after the application of loads, of a viscoelastic sandwich plate. The more practically useful aspect is the buckling phenomenon. The present analysis also predicts the buckling phenomenon for $0 \leq t \leq \infty$. Figure 13 shows a graph (like the ones obtained in the present analysis) of a critical axial load for antisymmetric or symmetric mode of buckling, vs time t for $0 \leq t \leq \infty$ in a given problem. The curve ABC shows the critical value of the axial load $N_{cr}(t)$. The region above the curve ABC is unstable, and the region below ABC is stable.

Suppose at $t = 0$, the plate is subjected to a step load which is axial (in the x direction) and compressive. If the magnitude of this load is smaller than $N_{cr}(t = \infty)$ represented by the line D'B', the whole line D'B' falls into the stable region, and therefore the plate will not buckle for any time t, $0 \leq t \leq \infty$. If the magnitude of the load is greater than $N_{cr}(t = 0)$, the line D"B" representing this case is completely in the unstable region,

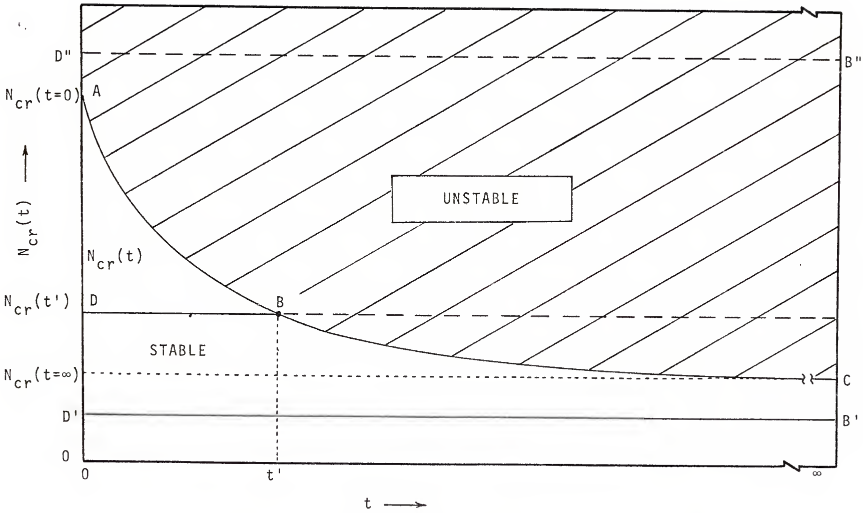


Figure 13. Applicability of the analysis of buckling

and the plate buckles immediately after the application of the load. Finally, if the magnitude of the load is greater than $N_{cr}(t = \infty)$ but less than $N_{cr}(t = 0)$, this is shown by the line DB which is in the stable region for $0 \leq t < t'$, and in the unstable region for $t' < t \leq \infty$. Therefore buckling occurs at $t = t'$.

Therefore an important prediction can be made that the plate will buckle at $t = t'$ after application of the load. Conversely, if the plate is required to stand a load for only a given length of time, say $t'$, the corresponding maximum load $N = N_{cr}(t = t')$ can be calculated in a reverse way.

## Conclusions

The analysis presented in this dissertation consists of two main parts connected by a comparatively short but important link in the middle. The first of the main parts is a nonlinear theory of elastic sandwich plates derived in a unified approach from the variational principle of Hellinger and Reissner in three-dimensional elasticity. It can be considered to be an extension of the earlier works of Ebcioglu and Habip. Ebcioglu considered a more general problem (of a sandwich plate), whereas Habip considered a single layer plate. Habip used a more general procedure (from the static and quasi-static elasticity point of view) viz., the Modified Hellinger-Reissner principle, than Hamilton's principle which was used by Ebcioglu. The present analysis has used Habip's procedure for the problem considered by Ebcioglu. Kim's linear analysis for the problem of a sandwich plate subjected to combined axial and transverse loads has been derived as a special case of the present analysis.

The second of the two main parts of this dissertation is the solution of the problem of a viscoelastic sandwich plate subjected to combined axial and transverse loads. This part is an extension and development of earlier analyses of viscoelastic sandwich plates by other authors. The present analysis has used a different procedure (the principle of correspondence) than those used by others, and has introduced a new idea of time dependent critical loads.

The main two parts of the dissertation described above are connected by the well-known correspondence principle in linear viscoelasticity, which has been described as the "short but important link" in the beginning of this section. The viscoelastic analysis has been derived from the elastic analysis using the correspondence principle.

Further extensions of the present work can be carried out on many fronts in both the fields of elasticity and viscoelasticity. For elastic sandwich plates, different and more general types of loading and boundary conditions can be considered. For viscoelastic sandwich plates, a four-parameter model or a still better model can be used. Also, nonlinear problems of viscoelastic plates can be considered using the governing equations of nonlinear viscoelasticity. Finally, on the numerical side, there are and will always be plenty of useful problems on sandwich plates.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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